IMC 2003

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by Peter, sqrtX, caugust

Day 1

1 (a) Let $a_1, a_2, ...$ be a sequenceof reals with $a_1 = 1$ and $a_{n+1} > \frac{3}{2}a_n$ for all n. Prove that $\lim_{n\to\infty} \frac{a_n}{\left(\frac{3}{2}\right)^{n-1}}$ exists. (finite or infinite)

(b) Prove that for all $\alpha > 1$ there is a sequence $a_1, a_2, ...$ with the same properties such that $\lim_{n\to\infty} \frac{a_n}{\left(\frac{3}{2}\right)^{n-1}} = \alpha$

2 Let $a_1, a_2, ..., a_{51}$ be non-zero elements of a field of characteristic p. We simultaneously replace each element with the sum of the 50 remaining ones. In this way we get a sequence $b_1, ..., b_{51}$. If this new sequence is a permutation of the original one, find all possible values of p.

3 Let $A \in \mathbb{R}^{n\times n}$ such that $3A^3 = A^2 + A + I$. Show that the sequence A^k converges to an idempotent matrix. (idempotent: $B^2 = B$)

4 Determine the set of all pairs (a,b) of positive integers for which the set of positive integers can be decomposed into 2 sets A and B so that $a \cdot A = b \cdot B$.

5 Let $g : [0, 1] \to \mathbb{R}$ be a continuous function and let $f_n : [0, 1] \to \mathbb{R}$ be a sequence of functions defined by $f_0(x) = g(x)$ and

$$f_{n+1}(x) = \frac{1}{x} \int_0^x f_n(t)dt.$$

Determine $\lim_{n\to\infty} f_n(x)$ for every $x \in (0, 1]$.

6 Let $p = \sum_{k=0}^{n} a_k X^k \in R[X]$ a polynomial such that all his roots lie in the half plane $\{z \in C | Re(z) < 0\}$. Prove that $a_k a_{k+3} < a_{k+1} a_{k+2}$, for every k=0,1,2...,n-3.

Day 2

1 Let $A, B \in \mathbb{R}^{n\times n}$ such that $AB + B + A = 0$. Prove that $AB = BA$.

2 Evaluate $\lim_{x\to 0^+} \int_x^{2x} \frac{\sin^m(t)}{t^n} dt$. $(m, n \in \mathbb{N})$

3 Let A be a closed subset of $\mathbb{R}^n$ and let B be the set of all those points $b \in \mathbb{R}^n$ for which there exists exactly one point $a_0 \in A$ such that $|a_0 - b| = \inf_{a\in A} |a-b|$.
Prove that B is dense in $\mathbb{R}^n$; that is, the closure of B is $\mathbb{R}^n$

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4 Find all the positive integers n for which there exists a Family $\mathcal{F}$ of three-element subsets of $S = \{1, 2, ..., n\}$ satisfying

(i) for any two different elements $a, b \in S$ there exists exactly one $A \in \mathcal{F}$ containing both a and b;

(ii) if a, b, c, x, y, z are elements of S such that $\{a, b, x\}, \{a, c, y\}, \{b, c, z\} \in \mathcal{F}$, then $\{x, y, z\} \in \mathcal{F}$.

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5 a) Show that for each function $f : \mathbb{Q} \times \mathbb{Q} \to \mathbb{R}$, there exists a function $g : \mathbb{Q} \to \mathbb{R}$ with $f(x, y) \le g(x) + g(y)$ for all $x, y \in \mathbb{Q}$.
b) Find a function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, for which there is no function $g : \mathbb{Q} \to \mathbb{R}$ such that $f(x, y) \le g(x) + g(y)$ for all $x, y \in \mathbb{R}$.

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6 Let (a_n) be the sequence defined by $a_0 = 1, a_{n+1} = \sum_{k=0}^{n} \dfrac{a_k}{n-k+2}$.
Find the limit

$$\lim_{n\to\infty} \sum_{k=0}^{n} \frac{a_k}{2^k},$$

if it exists.
