IMC 2004

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Day 1

1 Let S be an infinite set of real numbers such that $|x_1+x_2+\cdots+x_n|\leq 1$ for all finite subsets $\{x_1,x_2,\ldots,x_n\}\subset S$. Show that S is countable.

2 Let $f_1(x)=x^2-1$, and for each positive integer $n\geq 2$ define $f_n(x)=f_{n-1}(f_1(x))$. How many distinct real roots does the polynomial f_{2004} have?

3 Let A_n be the set of all the sums $\displaystyle\sum_{k=1}^n \arcsin x_k$, where $n\geq 2$, $x_k\in[0,1]$, and $\displaystyle\sum_{k=1}^n x_k=1$.

a) Prove that A_n is an interval.

b) Let a_n be the length of the interval A_n. Compute $\displaystyle\lim_{n\to\infty} a_n$.

4 Suppose $n\geq 4$ and let S be a finite set of points in the space ($\mathbb{R}^3$), no four of which lie in a plane. Assume that the points in S can be colored with red and blue such that any sphere which intersects S in at least 4 points has the property that exactly half of the points in the intersection of S and the sphere are blue. Prove that all the points of S lie on a sphere.

5 Let S be a set of $\displaystyle\binom{2n}{n}+1$ real numbers, where n is an positive integer. Prove that there exists a monotone sequence $\{a_i\}_{1\leq i\leq n+2}\subset S$ such that

$$|x_{i+1}-x_1|\geq 2|x_i-x_1|,$$

for all $i=2,3,\ldots,n+1$.

6 For every complex number z different from 0 and 1 we define the following function

$$f(z):=\sum\frac{1}{\log^4 z}$$

where the sum is over all branches of the complex logarithm.

a) Prove that there are two polynomials P and Q such that $f(z)=\dfrac{P(z)}{Q(z)}$ for all $z\in\mathbb{C}-\{0,1\}$.

b) Prove that for all $z \in \mathbb{C} - \{0,1\}$ we have

$$f(z) = \frac{z^3 + 4z^2 + z}{6(z-1)^4}.$$

Day 2

1 Let A be a real 4×2 matrix and B be a real 2×4 matrix such that

$$AB = \left(\begin{array}{cccc} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{array} \right).$$

Find BA.

2 Let $f, g : [a,b] \to [0,\infty)$ be two continuous and non-decreasing functions such that each $x \in [a,b]$ we have

$$\int_a^x \sqrt{f(t)}\, dt \le \int_a^x \sqrt{g(t)}\, dt \text{ and } \int_a^b \sqrt{f(t)}\, dt = \int_a^b \sqrt{g(t)}\, dt.$$

Prove that

$$\int_a^b \sqrt{1+f(t)}\, dt \ge \int_a^b \sqrt{1+g(t)}\, dt.$$

3 Let D be the closed unit disk in the plane, and let $z_1, z_2, \ldots, z_n$ be fixed points in D. Prove that there exists a point z in D such that the sum of the distances from z to each of the n points is greater or equal than n.

4 For $n \ge 1$ let M be an $n \times n$ complex array with distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$, with multiplicities $m_1, m_2, \ldots, m_k$ respectively. Consider the linear operator L_M defined by $L_M X = MX + XM^T$, for any complex $n \times n$ array X. Find its eigenvalues and their multiplicities. (M^T denotes the transpose matrix of M).

5 Prove that

$$\int_0^1 \int_0^1 \frac{dx\, dy}{\frac{1}{x} + |\log y| - 1} \le 1.$$

6 For $n \ge 0$ define the matrices A_n and B_n as follows: $A_0 = B_0 = (1)$, and for every $n > 0$ let

$$A_n = \left(\begin{array}{cc} A_{n-1} & A_{n-1} \\ A_{n-1} & B_{n-1} \end{array} \right) \text{ and } B_n = \left(\begin{array}{cc} A_{n-1} & A_{n-1} \\ A_{n-1} & 0 \end{array} \right).$$

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Denote by $S(M)$ the sum of all the elements of a matrix M. Prove that $S(A_n^{k-1}) = S(A_k^{n-1})$, for all $n, k \geq 2$.

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