IMC 2005

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Day 1

1	Let A be a $n \times n$ matrix such that $A_{ij} = i + j$. Find the rank of A. Not asked in the contest: A is diagonalisable since real symetric matrix it is not difficult to find its eigenvalues.
2	2) all elements in 0,1,2; B[n] = number of rows with no 2 sequent 0's; A[n] with no 3 sequent elements the same; prove —A[n+1]—=3.—B[n]—
3	3) f cont diff, $R \to]0,+\infty[$, prove $\lvert \int_0^1 f^3 - f(0)^2 \int_0^1 f \rvert \leq \max_{[0,1]} \lvert f' \rvert (\int_0^1 f)^2$
4	4) find all polynom with coeffs a permutation of $[1,...,n]$ and all roots rational
5	5) f twice cont diff, $\lvert f''(x) + 2xf'(x) + (x^2+1)f(x) \rvert \leq 1$. prove $\lim_{x \to +\infty} f(x) = 0$
6	6) G group, G_m and G_n commutative subgroups being the m and n th powers of the elements in G. Prove $G_{gcd(m,n)}$ is commutative.

Day 2

1	1. Let $f(x) = x^2 + bx + c$, M = x — —f(x)—¡1. Prove $\lvert M \rvert \leq 2\sqrt{2}$ (—...— = length of interval(s))
2	Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $(f(x))^n$ is a polynomial for every integer $n \geq 2$. Is f also a polynomial?
3	What is the maximal dimension of a linear subspace V of the vector space of real $n \times n$ matrices such that for all A in B in V, we have trace $(AB) = 0$?
4	Let $f : \mathbb{R} \to \mathbb{R}$ be a three times differentiable function. Prove that there exists $w \in [-1,1]$ such that $$\frac{f'''(w)}{6} = \frac{f(1)}{2} - \frac{f(-1)}{2} - f'(0).$$
5	Find all $r > 0$ such that when $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable, $\lVert \text{grad } f(0,0) \rVert = 1$, $\lVert \text{grad } f(u) - \text{grad } f(v) \rVert \leq \lVert u - v \rVert$, then the max of f on the disk $\lVert u \rVert \leq r$, is attained at exactly one point.

6 6. If p, q are rationals, $r = p + \sqrt{7}q$, then prove there exists a matrix $\left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \in M_2(Z) - (\pm I_2)$ for which $\frac{ar+b}{cr+d} = r$ and $det(A) = 1$