IMC 2006

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Day 1

1 Let $f:\mathbb{R}\to\mathbb{R}$ be a real function. Prove or disprove each of the following statements.
(a) If f is continuous and range(f)=$\mathbb{R}$ then f is monotonic
(b) If f is monotonic and range(f)=$\mathbb{R}$ then f is continuous
(c) If f is monotonic and f is continuous then range(f)=$\mathbb{R}$

2 Find the number of positive integers x satisfying the following two conditions:
1. $x<10^{2006}$
2. x^2-x is divisible by 10^{2006}

3 Let A be an nxn matrix with integer entries and $b_1,b_2,...,b_k$ be integers satisfying $detA=b_1\cdot b_2\cdot ...\cdot b_k$. Prove that there exist nxn-matrices $B_1,B_2,...,B_k$ with integers entries such that $A=B_1\cdot B_2\cdot ...\cdot B_k$ and $detB_i=b_i$ for all $i=1,...,k$.

4 Let f be a rational function (i.e. the quotient of two real polynomials) and suppose that $f(n)$ is an integer for infinitely many integers n. Prove that f is a polynomial.

5 Let a,b,c,d three strictly positive real numbers such that

$$a^2+b^2+c^2=d^2+e^2,$$

$$a^4+b^4+c^4=d^4+e^4.$$

Compare

$$a^3+b^3+c^3$$

with

$$d^3+e^3,$$

6 Find all sequences $a_0,a_1,\ldots,a_n$ of real numbers such that $a_n\neq 0$, for which the following statement is true:

If $f:\mathbb{R}\to\mathbb{R}$ is an n times differentiable function
and $x_0<x_1<\ldots<x_n$ are real numbers such that
$f(x_0)=f(x_1)=\ldots=f(x_n)=0$ then there is $h\in(x_0,x_n)$ for which

$$a_0f(h)+a_1f'(h)+\ldots+a_nf^{(n)}(h)=0.$$

Day 2

1 Let V be a convex polygon.
(a) Show that if V has $3k$ vertices, then V can be triangulated such that each vertex is in an odd number of triangles.
(b) Show that if the number of vertices is not divisible with 3, then V can be triangulated such that exactly 2 vertices have an even number of triangles.

2 Find all functions $f:\mathbb{R}\to R$ such that for any $a<b$, $f([a,b])$ is an interval of length $b-a$

3 Compare $\tan(\sin x)$ with $\sin(\tan x)$, for $x\in \left]0,\frac{\pi}{2}\right[$.

4 Let v_0 be the zero ector and let $v_1,...,v_{n+1}\in\mathbb{R}^n$ such that the Euclidian norm $|v_i-v_j|$ is rational for all $0\le i,j\le n+1$. Prove that $v_1,...,v_{n+1}$ are linearly dependent over the rationals.

5 Show that there are an infinity of integer numbers m,n, with $gcd(m,n)=1$ such that the equation $(x+m)^3=nx$ has 3 different integer sollutions.

6 The scores of this problem were:
one time 17/20 (by the runner-up)
one time 4/20 (by Andrei Negut)
one time 1/20 (by the winner)
the rest had zero... just to give an idea of the difficulty.

Let A_i,B_i,S_i ($i=1,2,3$) be invertible real 2×2 matrices such that -not all A_i have a common real eigenvector, -$A_i=S_i^{-1}B_iS_i$ for $i=1,2,3$, -$A_1A_2A_3=B_1B_2B_3=I$. Prove that there is an invertible 2×2 matrix S such that $A_i=S^{-1}B_iS$ for all $i=1,2,3$.