IMC 2007

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Day 1 August 5th

1 Let f be a polynomial of degree 2 with integer coefficients. Suppose that $f(k)$ is divisible by 5 for every integer k. Prove that all coefficients of f are divisible by 5.

2 Let $n \geq 2$ be an integer. What is the minimal and maximal possible rank of an $n \times n$ matrix whose n^2 entries are precisely the numbers $1, 2, \ldots, n^2$?

3 Call a polynomial $P(x_1, \ldots, x_k)$ *good* if there exist 2×2 real matrices $A_1, \ldots, A_k$ such that $P(x_1, \ldots, x_k) = \det \left(\sum_{i=1}^{k} x_i A_i \right)$.

Find all values of k for which all homogeneous polynomials with k variables of degree 2 are good. (A polynomial is homogeneous if each term has the same total degree.)

4 Let G be a finite group. For arbitrary sets $U, V, W \subset G$, denote by N_{UVW} the number of triples $(x, y, z) \in U \times V \times W$ for which xyz is the unity .
Suppose that G is partitioned into three sets A, B and C (i.e. sets A, B, C are pairwise disjoint and $G = A \cup B \cup C$). Prove that $N_{ABC} = N_{CBA}$.

5 Let n be a positive integer and $a_1, \ldots, a_n$ be arbitrary integers. Suppose that a function $f : \mathbb{Z} \to \mathbb{R}$ satisfies $\sum_{i=1}^{n} f(k + a_i l) = 0$ whenever k and l are integers and $l \neq 0$. Prove that $f = 0$.

6 How many nonzero coefficients can a polynomial $P(x)$ have if its coefficients are integers and $|P(z)| \leq 2$ for any complex number z of unit length?

Day 2 August 6th

1 Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Suppose that for any $c > 0$, the graph of f can be moved to the graph of cf using only a translation or a rotation. Does this imply that $f(x) = ax + b$ for some real numbers a and b?

2 Let x, y and z be integers such that $S = x^4 + y^4 + z^4$ is divisible by 29. Show that S is divisible by 29^4.

3 Let C be a nonempty closed bounded subset of the real line and $f : C \to C$ be a nondecreasing continuous function. Show that there exists a point $p \in C$ such that $f(p) = p$.

(A set is closed if its complement is a union of open intervals. A function g is nondecreasing if $g(x) \leq g(y)$ for all $x \leq y$.)

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4 Let $n > 1$ be an odd positive integer and $A = (a_{ij})_{i,j=1..n}$ be the $n \times n$ matrix with

$$a_{ij} = \begin{cases} 2 & \text{if } i = j \\ 1 & \text{if } i - j \equiv \pm 2 \pmod n \\ 0 & \text{otherwise} \end{cases}.$$

Find $\det A$.

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5 For each positive integer k, find the smallest number n_k for which there exist real $n_k \times n_k$ matrices $A_1, A_2, \ldots, A_k$ such that all of the following conditions hold:

(1) $A_1^2 = A_2^2 = \ldots = A_k^2 = 0$,

(2) $A_iA_j = A_jA_i$ for all $1 \leq i, j \leq k$, and

(3) $A_1A_2 \ldots A_k \neq 0$.

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6 Let $f \neq 0$ be a polynomial with real coefficients. Define the sequence $f_0, f_1, f_2, \ldots$ of polynomials by $f_0 = f$ and $f_{n+1} = f_n + f_n'$ for every $n \geq 0$. Prove that there exists a number N such that for every $n \geq N$, all roots of f_n are real.
