IMC 2009

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Day 1

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1 Suppose that $f, g : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(r) \le g(r) \quad \forall r \in \mathbb{Q}$$

Does this imply $f(x) \le g(x) \quad \forall x \in \mathbb{R}$ if

(a)f and g are non-decreasing ?
(b)f and g are continuous?

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2 Let A, B, C be real square matrices of the same order, and suppose A is invertible. Prove that

$$(A-B)C = BA^{-1} \implies C(A-B) = A^{-1}B$$

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3 In a town every two residents who are not friends have a friend in common, and no one is a friend of everyone else. Let us number the residents from 1 to n and let a_i be the number of friends of the i^{th} resident. Suppose that

$$\sum_{i=1}^{n} a_i^2 = n^2 - n$$

Let k be the smallest number of residents (at least three) who can be seated at a round table in such a way that any two neighbors are friends. Determine all possible values of k.

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4 Let $p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$ be a complex polynomial. Suppose that $1 = c_0 \ge c_1 \ge \cdots \ge c_n \ge 0$ is a sequence of real numbers which form a convex sequence. (That is $2c_k \le c_{k-1} + c_{k+1}$ for every $k = 1, 2, \cdots, n-1$) and consider the polynomial

$$q(z) = c_0 a_0 + c_1 a_1 z + c_2 a_2 z^2 + \cdots + c_n a_n z^n$$

Prove that :

$$\max_{|z|\le 1} q(z) \le \max_{|z|\le 1} p(z)$$

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5 Let n be a positive integer. An $n-$ *simplex* in $\mathbb{R}^n$ is given by $n+1$ points $P_0, P_1, \cdots, P_n$, called its vertices, which do not all belong to the same hyperplane. For every n-simplex $\mathcal{S}$ we denote by $v(\mathcal{S})$ the volume of $\mathcal{S}$, and we write $C(\mathcal{S})$ for the center of the unique sphere containing all the vertices of $\mathcal{S}$.
Suppose that P is a point inside an n-simplex $\mathcal{S}$. Let $\mathcal{S}_i$ be the n-simplex obtained from $\mathcal{S}$ by replacing its i^{th} vertex by P. Prove that :

$$\sum_{j=0}^{n} v(\mathcal{S}_j)C(\mathcal{S}_j) = v(\mathcal{S})C(\mathcal{S})$$

Day 2

1 Let ℓ be a line and P be a point in $\mathbb{R}^3$. Let S be the set of points X such that the distance from X to ℓ is greater than or equal to two times the distance from X to P. If the distance from P to ℓ is $d>0$, find Volume(S).

2 Suppose $f:\mathbb{R}\to\mathbb{R}$ is a two times differentiable function satisfying $f(0)=1, f'(0)=0$ and for all $x\in[0,\infty)$, it satisfies

$$f''(x)-5f'(x)+6f(x)\geq 0$$

Prove that, for all $x\in[0,\infty)$,

$$f(x)\geq 3e^{2x}-2e^{3x}$$

3 Let $A, B \in \mathcal{M}_n(\mathbb{C})$ be two $n\times n$ matrices such that

$$A^2B+BA^2=2ABA$$

Prove there exists $k\in\mathbb{N}$ such that

$$(AB-BA)^k=\mathbf{0}_n$$

Here $\mathbf{0}_n$ is the null matrix of order n.

4 Let p be a prime number and $\mathbf{W}\subseteq\mathbb{F}_p[x]$ be the smallest set satisfying the following :

(a) $x+1\in\mathbf{W}$ and $x^{p-2}+x^{p-3}+\cdots+x^2+2x+1\in\mathbf{W}$
(b) For γ_1,γ_2 in $\mathbf{W}$, we also have $\gamma(x)\in\mathbf{W}$, where $\gamma(x)$ is the remainder $(\gamma_1\circ\gamma_2)(x) \pmod{x^p-x}$.
How many polynomials are in $\mathbf{W}$?

5 Let $\mathbb{M}$ be the vector space of $m\times p$ real matrices. For a vector subspace $S\subseteq\mathbb{M}$, denote by $\delta(S)$ the dimension of the vector space generated by all columns of all matrices in S.
Say that a vector subspace $T\subseteq\mathbb{M}$ is a *covering matrix space* if

$$\bigcup_{A\in T, A\neq\mathbf{0}} \ker A = \mathbb{R}^p$$

Such a T is minimal if it doesn't contain a proper vector subspace $S \subset T$ such that S is also a covering matrix space.

(a) (8 points) Let T be a minimal covering matrix space and let $n = \dim(T)$
Prove that

$$\delta(T) \leq \binom{n}{2}$$

(b) (2 points) Prove that for every integer n we can find m and p, and a minimal covering matrix space T as above such that $\dim T = n$ and $\delta(T) = \binom{n}{2}$
