IMC 2010

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Day 1 July 26th

1 Let $0 < a < b$. Prove that $\int_a^b (x^2+1)e^{-x^2}dx \geq e^{-a^2} - e^{-b^2}$.

2 Compute the sum of the series $\sum_{k=0}^{\infty} \frac{1}{(4k+1)(4k+2)(4k+3)(4k+4)} = \frac{1}{1\cdot 2\cdot 3\cdot 4} + \frac{1}{5\cdot 6\cdot 7\cdot 8} + ...$

3 Define the sequence $x_1, x_2, ...$ inductively by $x_1 = \sqrt{5}$ and $x_{n+1} = x_n^2 - 2$ for each $n \geq 1$. Compute $\lim_{n\to\infty} \frac{x_1\cdot x_2\cdot x_3\cdot\ldots\cdot x_n}{x_{n+1}}$.

4 Let a, b be two integers and suppose that n is a positive integer for which the set $\mathbb{Z}\backslash\{ax^n + by^n \mid x, y \in \mathbb{Z}\}$ is finite. Prove that $n = 1$.

5 Suppose that a, b, c are real numbers in the interval $[-1, 1]$ such that $1 + 2abc \geq a^2 + b^2 + c^2$. Prove that $1 + 2(abc)^n \geq a^{2n} + b^{2n} + c^{2n}$ for all positive integers n.

Day 2 July 27th

1 (a) A sequence $x_1, x_2, \ldots$ of real numbers satisfies

$$x_{n+1} = x_n \cos x_n \text{ for all } n \geq 1.$$

Does it follows that this sequence converges for all initial values x_1? (5 points)

(b) A sequence $y_1, y_2, \ldots$ of real numbers satisfies

$$y_{n+1} = y_n \sin y_n \text{ for all } n \geq 1.$$

Does it follows that this sequence converges for all initial values y_1? (5 points)

2 Let $a_0, a_1, \ldots, a_n$ be positive real numbers such that $a_{k+1} - a_k \geq 1$ for all $k = 0, 1, \ldots, n-1$. Prove that

$$1 + \frac{1}{a_0}\left(1 + \frac{1}{a_1 - a_0}\right)\cdots\left(1 + \frac{1}{a_n - a_0}\right) \leq \left(1 + \frac{1}{a_0}\right)\left(1 + \frac{1}{a_1}\right)\cdots\left(1 + \frac{1}{a_n}\right).$$

3 Denote by S_n the group of permutations of the sequence $(1, 2, \dots, n)$. Suppose that G is a subgroup of S_n, such that for every $\pi \in G \setminus \{e\}$ there exists a unique $k \in \{1, 2, \dots, n\}$ for which $\pi(k) = k$. (Here e is the unit element of the group S_n.) Show that this k is the same for all $\pi \in G \setminus \{e\}$.

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4 Let A be a symmetric $m \times m$ matrix over the two-element field all of whose diagonal entries are zero. Prove that for every positive integer n each column of the matrix A^n has a zero entry.

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5 Suppose that for a function $f : \mathbb{R} \to \mathbb{R}$ and real numbers $a < b$ one has $f(x) = 0$ for all $x \in (a, b)$. Prove that $f(x) = 0$ for all $x \in \mathbb{R}$ if

$$\sum_{k=0}^{p-1} f\left(y + \frac{k}{p}\right) = 0$$

for every prime number p and every real number y.
