IMC 2011

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by rustam, m.candales

Day 1 July 30th

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1 Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. A point x is called a *shadow* point if there exists a point $y \in \mathbb{R}$ with $y > x$ such that $f(y) > f(x)$. Let $a < b$ be real numbers and suppose that
• all the points of the open interval $I = (a, b)$ are shadow points;
• a and b are not shadow points.
Prove that
a) $f(x) \le f(b)$ for all $a < x < b$;
b) $f(a) = f(b)$.

Proposed by Jos Luis Daz-Barrero, Barcelona

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2 Does there exist a real 3×3 matrix A such that tr$(A) = 0$ and $A^2 + A^t = I$? (tr(A) denotes the trace of A, A^t the transpose of A, and I is the identity matrix.)

Proposed by Moubinool Omarjee, Paris

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3 Let p be a prime number. Call a positive integer n interesting if

$$x^n - 1 = (x^p - x + 1)f(x) + pg(x)$$

for some polynomials f and g with integer coefficients.
a) Prove that the number $p^p - 1$ is interesting.
b) For which p is $p^p - 1$ the minimal interesting number?

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4 Let $A_1, A_2, \dots, A_n$ be finite, nonempty sets. Define the function

$$f(t) = \sum_{k=1}^{n} \sum_{1 \le i_1 < i_2 < \dots < i_k \le n} (-1)^{k-1} t^{|A_{i_1} \cup A_{i_2} \cup \dots \cup A_{i_k}|}.$$

Prove that f is nondecreasing on $[0, 1]$.
($|A|$ denotes the number of elements in A.)

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5 Let n be a positive integer and let V be a $(2n-1)$-dimensional vector space over the two-element field. Prove that for arbitrary vectors $v_1, \dots, v_{4n-1} \in V$, there exists a sequence $1 \le i_1 < \dots < i_{2n} \le 4n-1$ of indices such that $v_{i_1} + \dots + v_{i_{2n}} = 0$.

Day 2 July 31st

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1 Let $(a_n) \subset (\frac{1}{2}, 1)$. Define the sequence $x_0 = 0, x_{n+1} = \dfrac{a_{n+1} + x_n}{1 + a_{n+1}x_n}$. Is this sequence convergent? If yes find the limit.

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2 An alien race has three genders: male, female and emale. A married triple consists of three persons, one from each gender who all like each other. Any person is allowed to belong to at most one married triple. The feelings are always mutual (if x likes y then y likes x).
The race wants to colonize a planet and sends n males, n females and n emales. Every expedition member likes at least k persons of each of the two other genders. The problem is to create as many married triples so that the colony could grow.

a) Prove that if n is even and $k \geq 1/2$ then there might be no married triple.
b) Prove that if $k \geq 3n/4$ then there can be formed n married triple (i.e. everybody is in a triple).

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3 Calculate $\displaystyle\sum_{n=1}^{\infty} \ln\left(1 + \frac{1}{n}\right) \ln\left(1 + \frac{1}{2n}\right) \ln\left(1 + \frac{1}{2n+1}\right)$.

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4 Let f be a polynomial with real coefficients of degree n. Suppose that $\dfrac{f(x) - f(y)}{x - y}$ is an integer for all $0 \leq x < y \leq n$. Prove that $a - b | f(a) - f(b)$ for all distinct integers a, b.

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5 Let $F = A_0A_1...A_n$ be a convex polygon in the plane. Define for all $1 \leq k \leq n-1$ the operation f_k which replaces F with a new polygon $f_k(F) = A_0A_1..A_{k-1}A'_kA_{k+1}...A_n$ where A'_k is the symmetric of A_k with respect to the perpendicular bisector of $A_{k-1}A_{k+1}$. Prove that $(f_1 \circ f_2 \circ f_3 \circ ... \circ f_{n-1})^n(F) = F$.
