IMC 2013
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by Eukleidis

Day 1 July 28th

1 Let A and B be real symmetric matrixes with all eigenvalues strictly greater than 1. Let λ be a real eigenvalue of matrix AB. Prove that $|\lambda| > 1$.

Proposed by Pavel Kozhevnikov, MIPT, Moscow.

2 Let $f:\mathcal{R}\to\mathcal{R}$ be a twice differentiable function. Suppose $f\left(0\right)=0$. Prove there exists $\xi\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ such that

$$f''\left(\xi\right)=f\left(\xi\right)\left(1+2\tan^2\xi\right).$$

Proposed by Karen Keryan, Yerevan State University, Yerevan, Armenia.

3 There are $2n$ students in a school $(n\in\mathbb{N}, n\geqslant 2)$. Each week n students go on a trip. After several trips the following condition was fulfiled: every two students were together on at least one trip. What is the minimum number of trips needed for this to happen?

Proposed by Oleksandr Rybak, Kiev, Ukraine.

4 Let $n\geqslant 3$ and let $x_1, x_2, ..., x_n$ be nonnegative real numbers. Define $A=\sum_{i=1}^n x_i, B=\sum_{i=1}^n x_i^2, C=\sum_{i=1}^n x_i^3$. Prove that:

$$\left(n+1\right)A^2B+\left(n-2\right)B^2\geqslant A^4+\left(2n-2\right)AC.$$

Proposed by Gza Ks, Etvs University, Budapest.

5 Does there exist a sequence (a_n) of complex numbers such that for every positive integer p we have that $\sum_{n=1}^{+\infty} a_n^p$ converges if and only if p is not a prime?

Proposed by Tom Brta, Charles University, Prague.

Day 2 August 9th

1 Let z be a complex number with $|z+1|>2$. Prove that $\left|z^3+1\right|>1$.

Proposed by Walther Janous and Gerhard Kirchner, Innsbruck.

2 Let p, q be relatively prime positive integers. Prove that

$$\sum_{k=0}^{pq-1}(-1)^{\left\lfloor \frac{k}{p}\right\rfloor+\left\lfloor \frac{k}{q}\right\rfloor}=\begin{cases}0 & \text{if } pq \text{ is even}\\ 1 & \text{if } pq \text{ odd}\end{cases}$$

Proposed by Alexander Bolbot, State University, Novosibirsk.

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3 Suppose that $v_1, v_2, ..., v_d$ are unit vectors in $\mathbb{R}^d$. Prove that there exists a unitary vector u such that $|u \cdot v_i| \le \frac{1}{\sqrt{d}}$ for $i = 1, 2, ..., d$.

Note. Here $\cdot$ denotes the usual scalar product on $\mathbb{R}^d$.

Proposed by Tomasz Tkocz, University of Warwick.

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4 Does there exist an infinite set M consisting of positive integers such that for any $a, b \in M$ with $a < b$ the sum $a + b$ is square-free?
Note. A positive integer is called square-free if no perfect square greater than 1 divides it.

Proposed by Fedor Petrov, St. Petersburg State University.

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5 Consider a circular necklace with 2013 beads. Each bead can be paintes either green or white. A painting of the necklace is called *good* if among any 21 successive beads there is at least one green bead. Prove that the number of good paintings of the necklace is odd.

Note. Two paintings that differ on some beads, but can be obtained from each other by rotating or flipping the necklace, are counted as different paintings.

Proposed by Vsevolod Bykov and Oleksandr Rybak, Kiev.
