Online Math Open Problems 2012

www.artofproblemsolving.com/community/c4386

by Zhero, math154

–	Winter
1	The average of two positive real numbers is equal to their difference. What is the ratio of the larger number to the smaller one? *Author: Ray Li*
2	How many ways are there to arrange the letters A, A, A, H, H in a row so that the sequence HA appears at least once? *Author: Ray Li*
3	A lucky number is a number whose digits are only 4 or 7. What is the 17th smallest lucky number? *Author: Ray Li* -Lucky numbers are positive. -"only 4 or 7" includes combinations of 4 and 7, as well as only 4 and only 7. That is, 4 and 47 are both lucky numbers.
4	How many positive even numbers have an even number of digits and are less than 10000? *Author: Ray Li*
5	Congruent circles Γ_1 and Γ_2 have radius $2012,$ and the center of Γ_1 lies on Γ_2. Suppose that Γ_1 and Γ_2 intersect at A and B. The line through A perpendicular to AB meets Γ_1 and Γ_2 again at C and D, respectively. Find the length of CD. *Author: Ray Li*
6	Alice's favorite number has the following properties: - It has 8 distinct digits. -The digits are decreasing when read from left to right. -It is divisible by 180. What is Alice's favorite number? *Author: Anderson Wang*

7 A board 64 inches long and 4 inches high is inclined so that the long side of the board makes a 30 degree angle with the ground. The distance from the ground to the highest point on the board can be expressed in the form $a + b\sqrt{c}$ where a, b, c are positive integers and c is not divisible by the square of any prime. What is $a + b + c$?

Author: Ray Li

The problem is intended to be a two-dimensional problem. The board's dimensions are 64 by 4. The long side of the board makes a 30 degree angle with the ground. One corner of the board is touching the ground.

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8 An $8\times8\times8$ cube is painted red on 3 faces and blue on 3 faces such that no corner is surrounded by three faces of the same color. The cube is then cut into 512 unit cubes. How many of these cubes contain both red and blue paint on at least one of their faces?

Author: Ray Li

The problem asks for the number of cubes that contain red paint on at least one face and blue paint on at least one other face, not for the number of cubes that have both colors of paint on at least one face (which can't even happen.)

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9 At a certain grocery store, cookies may be bought in boxes of 10 or 21. What is the minimum positive number of cookies that must be bought so that the cookies may be split evenly among 13 people?

Author: Ray Li

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10 A drawer has 5 pairs of socks. Three socks are chosen at random. If the probability that there is a pair among the three is $\frac{m}{n}$, where m and n are relatively prime positive integers, what is $m + n$?

Author: Ray Li

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11 If

$$\frac{1}{x} + \frac{1}{2x^2} + \frac{1}{4x^3} + \frac{1}{8x^4} + \frac{1}{16x^5} + \cdots = \frac{1}{64},$$

and x can be expressed in the form $\frac{m}{n}$, where m, n are relatively prime positive integers, find $m + n$.

Author: Ray Li

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12 A *cross-pentomino* is a shape that consists of a unit square and four other unit squares each sharing a different edge with the first square. If a cross-pentomino is inscribed in a circle of radius R, what is $100R^2$?

Author: Ray Li

13 A circle ω has center O and radius r. A chord BC of ω also has length r, and the tangents to ω at B and C meet at A. Ray AO meets ω at D past O, and ray OA meets the circle centered at A with radius AB at E past A. Compute the degree measure of $\angle DBE$.

Author: Ray Li

14 Al told Bob that he was thinking of 2011 distinct positive integers. He also told Bob the sum of those 2011 distinct positive integers. From this information, Bob was able to determine all 2011 integers. How many possible sums could Al have told Bob?

Author: Ray Li

15 Five bricklayers working together finish a job in 3 hours. Working alone, each bricklayer takes at most 36 hours to finish the job. What is the smallest number of minutes it could take the fastest bricklayer to complete the job alone?

Author: Ray Li

16 Let $A_1B_1C_1D_1A_2B_2C_2D_2$ be a unit cube, with $A_1B_1C_1D_1$ and $A_2B_2C_2D_2$ opposite square faces, and let M be the center of face $A_2B_2C_2D_2$. Rectangular pyramid $MA_1B_1C_1D_1$ is cut out of the cube. If the surface area of the remaining solid can be expressed in the form $a+\sqrt{b}$, where a and b are positive integers and b is not divisible by the square of any prime, find $a+b$.

Author: Alex Zhu

17 Each pair of vertices of a regular 10-sided polygon is connected by a line segment. How many unordered pairs of distinct parallel line segments can be chosen from these segments?

Author: Ray Li

18 The sum of the squares of three positive numbers is 160. One of the numbers is equal to the sum of the other two. The difference between the smaller two numbers is 4. What is the difference between the cubes of the smaller two numbers?

Author: Ray Li

The problem should ask for the positive difference.

19 There are 20 geese numbered $1-20$ standing in a line. The even numbered geese are standing at the front in the order $2,4,\dots,20$, where 2 is at the front of the line. Then the odd numbered geese are standing behind them in the order, $1,3,5,\dots,19$, where 19 is at the end of the line. The geese want to rearrange themselves in order, so that they are ordered $1,2,\dots,20$ (1 is at the front), and they do this by successively swapping two adjacent geese. What is the minimum number of swaps required to achieve this formation?

Author: Ray Li

20 Let ABC be a right triangle with a right angle at C. Two lines, one parallel to AC and the other parallel to BC, intersect on the hypotenuse AB. The lines split the triangle into two triangles and a rectangle. The two triangles have areas 512 and 32. What is the area of the rectangle?

Author: Ray Li

21 If

$$2011^{2011^{2012}} = x^x$$

for some positive integer x, how many positive integer factors does x have?

Author: Alex Zhu

22 Find the largest prime number p such that when $2012!$ is written in base p, it has at least p trailing zeroes.

Author: Alex Zhu

23 Let ABC be an equilateral triangle with side length 1. This triangle is rotated by some angle about its center to form triangle DEF. The intersection of ABC and DEF is an equilateral hexagon with an area that is $\frac{4}{5}$ the area of ABC. The side length of this hexagon can be expressed in the form $\frac{m}{n}$ where m and n are relatively prime positive integers. What is $m+n$?

Author: Ray Li

24 Find the number of ordered pairs of positive integers (a, b) with $a+b$ prime, $1 \leq a, b \leq 100$, and $\frac{ab+1}{a+b}$ is an integer.

Author: Alex Zhu

25 Let a, b, c be the roots of the cubic x^3+3x^2+5x+7. Given that P is a cubic polynomial such that $P(a) = b+c$, $P(b) = c+a$, $P(c) = a+b$, and $P(a+b+c) = -16$, find $P(0)$.

Author: Alex Zhu

26 Xavier takes a permutation of the numbers 1 through 2011 at random, where each permutation has an equal probability of being selected. He then cuts the permutation into increasing contiguous subsequences, such that each subsequence is as long as possible. Compute the expected number of such subsequences.

Author: Alex Zhu

An increasing contiguous subsequence is an increasing subsequence all of whose terms are adjacent in the original sequence. For example, 1,3,4,5,2 has two maximal increasing contiguous subsequences: (1,3,4,5) and (2).

27 a and b are real numbers that satisfy

$$a^4 + a^2b^2 + b^4 = 900,$$

$$a^2 + ab + b^2 = 45.$$

Find the value of $2ab$.

Author: Ray Li

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28 A fly is being chased by three spiders on the edges of a regular octahedron. The fly has a speed of 50 meters per second, while each of the spiders has a speed of r meters per second. The spiders choose their starting positions, and choose the fly's starting position, with the requirement that the fly must begin at a vertex. Each bug knows the position of each other bug at all times, and the goal of the spiders is for at least one of them to catch the fly. What is the maximum c so that for any $r < c$, the fly can always avoid being caught?

Author: Anderson Wang

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29 How many positive integers a with $a \le 154$ are there such that the coefficient of x^a in the expansion of

$$(1 + x^7 + x^{14} + \cdots + x^{77})(1 + x^{11} + x^{22} + \cdots + x^{77})$$

is zero?

Author: Ray Li

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30 The Lattice Point Jumping Frog jumps between lattice points in a coordinate plane that are exactly 1 unit apart. The Lattice Point Jumping Frog starts at the origin and makes 8 jumps, ending at the origin. Additionally, it never lands on a point other than the origin more than once. How many possible paths could the frog have taken?

Author: Ray Li

-The Lattice Jumping Frog is allowed to visit the origin more than twice.
-The path of the Lattice Jumping Frog is an ordered path, that is, the order in which the Lattice Jumping Frog performs its jumps matters.

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31 Let ABC be a triangle inscribed in circle Γ, centered at O with radius 333. Let M be the midpoint of AB, N be the midpoint of AC, and D be the point where line AO intersects BC. Given that lines MN and BO concur on Γ and that $BC = 665$, find the length of segment AD.

Author: Alex Zhu

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32 The sequence $\{a_n\}$ satisfies $a_0 = 1, a_1 = 2011$, and $a_n = 2a_{n-1} + a_{n-2}$ for all $n \geq 2$. Let

$$S = \sum_{i=1}^{\infty} \frac{a_{i-1}}{a_i^2 - a_{i-1}^2}$$

What is $\frac{1}{S}$?

Author: Ray Li

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33 You are playing a game in which you have 3 envelopes, each containing a uniformly random amount of money between 0 and 1000 dollars. (That is, for any real $0 \leq a < b \leq 1000$, the probability that the amount of money in a given envelope is between a and b is $\frac{b-a}{1000}$.) At any step, you take an envelope and look at its contents. You may choose either to keep the envelope, at which point you finish, or discard it and repeat the process with one less envelope. If you play to optimize your expected winnings, your expected winnings will be E. What is $\lfloor E \rfloor$, the greatest integer less than or equal to E?

Author: Alex Zhu

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34 p, q, r are real numbers satisfying

$$\frac{(p+q)(q+r)(r+p)}{pqr} = 24$$

$$\frac{(p-2q)(q-2r)(r-2p)}{pqr} = 10.$$

Given that $\frac{p}{q} + \frac{q}{r} + \frac{r}{p}$ can be expressed in the form $\frac{m}{n}$, where m, n are relatively prime positive integers, compute $m + n$.

Author: Alex Zhu

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35 Let $s(n)$ be the number of 1's in the binary representation of n. Find the number of ordered pairs of integers (a, b) with $0 \leq a < 64, 0 \leq b < 64$ and $s(a+b) = s(a) + s(b) - 1$.

Author:Anderson Wang

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36 Let s_n be the number of solutions to $a_1 + a_2 + a_3 + a_4 + b_1 + b_2 = n$, where a_1, a_2, a_3 and a_4 are elements of the set $\{2, 3, 5, 7\}$ and b_1 and b_2 are elements of the set $\{1, 2, 3, 4\}$. Find the number of n for which s_n is odd.

Author: Alex Zhu

s_n is the number of *ordered* solutions $(a_1, a_2, a_3, a_4, b_1, b_2)$ to the equation, where each a_i lies in $\{2, 3, 5, 7\}$ and each b_i lies in $\{1, 2, 3, 4\}$.

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37 In triangle ABC, $AB = 1$ and $AC = 2$. Suppose there exists a point P in the interior of triangle ABC such that $\angle PBC = 70^\circ$, and that there are points E and D on segments AB and AC, such that $\angle BPE = \angle EPA = 75^\circ$ and $\angle APD = \angle DPC = 60^\circ$. Let BD meet CE at Q, and let AQ meet BC at F. If M is the midpoint of BC, compute the degree measure of $\angle MPF$.

Authors: Alex Zhu and Ray Li

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38 Let S denote the sum of the 2011th powers of the roots of the polynomial $(x-2^0)(x-2^1)\cdots(x-2^{2010}) - 1$. How many ones are in the binary expansion of S?

Author: Alex Zhu

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39 For positive integers n, let $\nu_3(n)$ denote the largest integer k such that 3^k divides n. Find the number of subsets S (possibly containing 0 or 1 elements) of $\{1, 2, \ldots, 81\}$ such that for any distinct $a, b \in S$, $\nu_3(a-b)$ is even.

Author: Alex Zhu

We only need $\nu_3(a-b)$ to be even for $a > b$.

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40 Suppose x, y, z, and w are positive reals such that

$$x^2 + y^2 - \frac{xy}{2} = w^2 + z^2 + \frac{wz}{2} = 36$$

$$xz + yw = 30.$$

Find the largest possible value of $(xy + wz)^2$.

Author: Alex Zhu

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41 Find the remainder when

$$\sum_{i=2}^{63} \frac{i^{2011} - i}{i^2 - 1}.$$

is divided by 2016.

Author: Alex Zhu

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42 In triangle ABC, $\sin \angle A = \frac{4}{5}$ and $\angle A < 90^\circ$ Let D be a point outside triangle ABC such that $\angle BAD = \angle DAC$ and $\angle BDC = 90^\circ$. Suppose that $AD = 1$ and that $\frac{BD}{CD} = \frac{3}{2}$. If $AB + AC$ can be expressed in the form $\frac{a\sqrt{b}}{c}$ where a, b, c are pairwise relatively prime integers, find $a + b + c$.

Author: Ray Li

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43 An integer x is selected at random between 1 and $2011!$ inclusive. The probability that $x^x - 1$ is divisible by 2011 can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find m.

Author: Alex Zhu

44 Given a set of points in space, a *jump* consists of taking two points, P and Q, and replacing P with the reflection of P over Q. Find the smallest number n such that for any set of n lattice points in 10-dimensional-space, it is possible to perform a finite number of jumps so that some two points coincide.

Author: Anderson Wang

45 Let K_1, K_2, K_3, K_4, K_5 be 5 distinguishable keys, and let D_1, D_2, D_3, D_4, D_5 be 5 distinguishable doors. For $1 \leq i \leq 5$, key K_i opens doors D_i and D_{i+1} (where $D_6 = D_1$) and can only be used once. The keys and doors are placed in some order along a hallway. Key$ha walks into the hallway, picks a key and opens a door with it, such that she never obtains a key before all the doors in front of it are unlocked. In how many orders can the keys and doors be placed such that Key$ha can open all of the doors?

Author: Mitchell Lee

-The doors and keys are in series. In other words, the doors aren't lined up along the side of the hallway. They are blocking Key$ha's path to the end, and the only way she can get past them is by getting the appropriate keys along the hallway.
-The doors and keys appear consecutively along the hallway. For example, she might find $K_1D_1K_2D_2K_3D_3K_4D_4K_5D_5$ down the hallway in that order. Also, by "she never obtains a key before all the doors in front of it are unlocked," we mean that she cannot obtain a key before all the doors appearing before the key are unlocked. In essence, it merely states that locked doors cannot be passed.
-The doors and keys do not need to alternate down the hallway.

46 If f is a function from the set of positive integers to itself such that $f(x) \leq x^2$ for all natural x, and $f\left(f(f(x))f(f(y))\right) = xy$ for all naturals x and y. Find the number of possible values of $f(30)$.

Author: Alex Zhu

47 Let $ABCD$ be an isosceles trapezoid with bases $AB = 5$ and $CD = 7$ and legs $BC = AD = 2\sqrt{10}$. A circle ω with center O passes through $A, B, C,$ and D. Let M be the midpoint of segment CD, and ray AM meet ω again at E. Let N be the midpoint of BE and P be the intersection of BE with CD. Let Q be the intersection of ray ON with ray DC. There is a point R on the circumcircle of PNQ such that $\angle PRC = 45^\circ$. The length of DR can be expressed in the form $\frac{m}{n}$ where m and n are relatively prime positive integers. What is $m + n$?

Author: Ray Li

48 Suppose that

$$\sum_{i=1}^{982} 7^{i^2}$$

can be expressed in the form $983q + r$, where q and r are integers and $0 \leq r \leq 492$. Find r.

Author: Alex Zhu

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49 Find the magnitude of the product of all complex numbers c such that the recurrence defined by $x_1 = 1, x_2 = c^2 - 4c + 7$, and $x_{n+1} = (c^2 - 2c)^2 x_n x_{n-1} + 2x_n - x_{n-1}$ also satisfies $x_{1006} = 2011$.

Author: Alex Zhu

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50 In tetrahedron $SABC$, the circumcircles of faces SAB, SBC, and SCA each have radius 108. The inscribed sphere of $SABC$, centered at I, has radius 35. Additionally, $SI = 125$. Let R be the largest possible value of the circumradius of face ABC. Given that R can be expressed in the form $\sqrt{\frac{m}{n}}$, where m and n are relatively prime positive integers, find $m + n$.

Author: Alex Zhu

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1 Calvin was asked to evaluate $37 + 31 \times a$ for some number a. Unfortunately, his paper was tilted 45 degrees, so he mistook multiplication for addition (and vice versa) and evaluated $37 \times 31 + a$ instead. Fortunately, Calvin still arrived at the correct answer while still following the order of operations. For what value of a could this have happened?

Ray Li.

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2 Petya gave Vasya a number puzzle. Petya chose a digit X and said, "I am thinking of a three digit number that is divisible by 11. The hundreds digit is X and the tens digit is 3. Find the units digit." Vasya was excited because he knew how to solve this problem, but then realized that the problem Petya gave did not have an answer. What digit X did Petya chose?

Ray Li.

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3 Darwin takes an 11×11 grid of lattice points and connects every pair of points that are 1 unit apart, creating a 10×10 grid of unit squares. If he never retraced any segment, what is the total length of all segments that he drew?

Ray Li.

-The problem asks for the total length of all *unit* segments (with two lattice points in the grid as endpoints) he drew.

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4 Let lcm(a, b) denote the least common multiple of a and b. Find the sum of all positive integers x such that $x \leq 100$ and lcm$(16, x) = 16x$.

Ray Li.

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5 Two circles have radius 5 and 26. The smaller circle passes through center of the larger one. What is the difference between the lengths of the longest and shortest chords of the larger circle that are tangent to the smaller circle?

Ray Li.

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6 An elephant writes a sequence of numbers on a board starting with 1. Each minute, it doubles the sum of all the numbers on the board so far, and without erasing anything, writes the result on the board. It stops after writing a number greater than one billion. How many distinct prime factors does the largest number on the board have?

Ray Li.

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7 Two distinct points A and B are chosen at random from 15 points equally spaced around a circle centered at O such that each pair of points A and B has the same probability of being chosen. The probability that the perpendicular bisectors of OA and OB intersect strictly inside the circle can be expressed in the form $\frac{m}{n}$, where m, n are relatively prime positive integers. Find $m+n$.

Ray Li.

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8 In triangle ABC let D be the foot of the altitude from A. Suppose that $AD = 4$, $BD = 3$, $CD = 2$, and AB is extended past B to a point E such that $BE = 5$. Determine the value of CE^2.

Ray Li.

-Triangle ABC is acute.

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9 Define a sequence of integers by $T_1 = 2$ and for $n \geq 2$, $T_n = 2^{T_{n-1}}$. Find the remainder when $T_1 + T_2 + \cdots + T_{256}$ is divided by 255.

Ray Li.

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10 There are 29 unit squares in the diagram below. A frog starts in one of the five (unit) squares on the top row. Each second, it hops either to the square directly below its current square (if that square exists), or to the square down one unit and left one unit of its current square (if that square exists), until it reaches the bottom. Before it reaches the bottom, it must make a hop every second. How many distinct paths (from the top row to the bottom row) can the frog take?

Ray Li.

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11 Let $ABCD$ be a rectangle. Circles with diameters AB and CD meet at points P and Q inside the rectangle such that P is closer to segment BC than Q. Let M and N be the midpoints of segments AB and CD. If $\angle MPN = 40^\circ$, find the degree measure of $\angle BPC$.

Ray Li.

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12 Let $a_1, a_2, \ldots$ be a sequence defined by $a_1 = 1$ and for $n \geq 1$, $a_{n+1} = \sqrt{a_n^2 - 2a_n + 3} + 1$. Find a_{513}.

Ray Li.

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13 A number is called *6-composite* if it has exactly 6 composite factors. What is the 6th smallest 6-composite number? (A number is *composite* if it has a factor not equal to 1 or itself. In particular, 1 is not composite.)

Ray Li.

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14 When Applejack begins to buck trees, she starts off with 100 energy. Every minute, she may either choose to buck n trees and lose 1 energy, where n is her current energy, or rest (i.e. buck 0 trees) and gain 1 energy. What is the maximum number of trees she can buck after 60 minutes have passed?

Anderson Wang.

-The problem asks for the maximum *total* number of trees she can buck in 60 minutes, not the maximum number she can buck on the 61st minute.
-She does not have an energy cap. In particular, her energy may go above 100 if, for instance, she chooses to rest during the first minute.

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15 How many sequences of nonnegative integers $a_1, a_2, \ldots, a_n$ ($n \geq 1$) are there such that $a_1 \cdot a_n > 0$, $a_1 + a_2 + \cdots + a_n = 10$, and $\prod_{i=1}^{n-1}(a_i + a_{i+1}) > 0$?

Ray Li.

-If you find the wording of the problem confusing, you can use the following, equivalent wording: "How many finite sequences of nonnegative integers are there such that (i) the sum of the elements is 10; (ii) the first and last elements are both positive; and (iii) among every pair of adjacent integers in the sequence, at least one is positive."

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16 Let ABC be a triangle with $AB = 4024$, $AC = 4024$, and $BC = 2012$. The reflection of line AC over line AB meets the circumcircle of $\triangle ABC$ at a point $D \neq A$. Find the length of segment CD.

Ray Li.

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17 Find the number of integers a with $1 \le a \le 2012$ for which there exist nonnegative integers x, y, z satisfying the equation

$$x^2(x^2+2z) - y^2(y^2+2z) = a.$$

Ray Li.

-x, y, z are not necessarily distinct.

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18 There are 32 people at a conference. Initially nobody at the conference knows the name of anyone else. The conference holds several 16-person meetings in succession, in which each person at the meeting learns (or relearns) the name of the other fifteen people. What is the minimum number of meetings needed until every person knows everyone else's name?

David Yang, Victor Wang.

See the "odd version" here (http://www.artofproblemsolving.com/Forum/viewtopic.php?f=810\&t=500914).

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19 In trapezoid $ABCD$, $AB < CD$, $AB \perp BC$, $AB \parallel CD$, and the diagonals AC, BD are perpendicular at point P. There is a point Q on ray CA past A such that $QD \perp DC$. If

$$\frac{QP}{AP} + \frac{AP}{QP} = \left(\frac{51}{14}\right)^4 - 2,$$

then $\frac{BP}{AP} - \frac{AP}{BP}$ can be expressed in the form $\frac{m}{n}$ for relatively prime positive integers m, n. Compute $m+n$.

Ray Li.

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20 The numbers $1, 2, \dots, 2012$ are written on a blackboard. Each minute, a student goes up to the board, chooses two numbers x and y, erases them, and writes the number $2x+2y$ on the board. This continues until only one number N remains. Find the remainder when the maximum possible value of N is divided by 1000.

Victor Wang.

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21 A game is played with 16 cards laid out in a row. Each card has a black side and a red side, and initially the face-up sides of the cards alternate black and red with the leftmost card black-side-up. A move consists of taking a consecutive sequence of cards (possibly only containing 1 card) with leftmost card black-side-up and the rest of the cards red-side-up, and flipping all of these cards over. The game ends when a move can no longer be made. What is the maximum possible number of moves that can be made before the game ends?

Ray Li.

See a close variant here (`http://www.artofproblemsolving.com/Forum/viewtopic.php?f=810\&t=500913`).

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22 Let $c_1, c_2, \dots, c_{6030}$ be 6030 real numbers. Suppose that for any 6030 real numbers $a_1, a_2, \dots, a_{6030}$, there exist 6030 real numbers $\{b_1, b_2, \dots, b_{6030}\}$ such that

$$a_n = \sum_{k=1}^{n} b_{\gcd(k,n)}$$

and

$$b_n = \sum_{d|n} c_d a_{n/d}$$

for $n = 1, 2, \dots, 6030$. Find c_{6030}.

Victor Wang.

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23 For reals $x \ge 3$, let $f(x)$ denote the function

$$f(x) = \frac{-x + x\sqrt{4x-3}}{2}.$$

Let $a_1, a_2, \dots$, be the sequence satisfying $a_1 > 3$, $a_{2013} = 2013$, and for $n = 1, 2, \dots, 2012$, $a_{n+1} = f(a_n)$. Determine the value of

$$a_1 + \sum_{i=1}^{2012} \frac{a_{i+1}^3}{a_i^2 + a_i a_{i+1} + a_{i+1}^2}.$$

Ray Li.

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24 In scalene $\triangle ABC$, I is the incenter, I_a is the A-excenter, D is the midpoint of arc BC of the circumcircle of ABC not containing A, and M is the midpoint of side BC. Extend ray IM past M to point P such that $IM = MP$. Let Q be the intersection of DP and MI_a, and R be the point on the line MI_a such that $AR \parallel DP$. Given that $\frac{AI_a}{AI} = 9$, the ratio $\frac{QM}{RI_a}$ can be expressed in the form $\frac{m}{n}$ for two relatively prime positive integers m, n. Compute $m + n$.

Ray Li.

-"Arc BC of the circumcircle" means "the arc with endpoints B and C not containing A".

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25 Suppose 2012 reals are selected independently and at random from the unit interval $[0, 1]$, and then written in nondecreasing order as $x_1 \le x_2 \le \dots \le x_{2012}$. If the probability that $x_{i+1} - x_i \le \frac{1}{2011}$ for $i = 1, 2, \dots, 2011$ can be expressed in the form $\frac{m}{n}$ for relatively prime positive integers m, n, find the remainder when $m + n$ is divided by 1000.

Victor Wang.

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26 Find the smallest positive integer k such that

$$\binom{x+kb}{12} \equiv \binom{x}{12} \pmod{b}$$

for all positive integers b and x. (*Note:* For integers a, b, c we say $a \equiv b \pmod{c}$ if and only if $a-b$ is divisible by c.)

Alex Zhu.

-$\binom{y}{12} = \frac{y(y-1)\cdots(y-11)}{12!}$ for all integers y. In particular, $\binom{y}{12} = 0$ for $y = 1, 2, \ldots, 11$.

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27 Let ABC be a triangle with circumcircle ω. Let the bisector of $\angle ABC$ meet segment AC at D and circle ω at $M \neq B$. The circumcircle of $\triangle BDC$ meets line AB at $E \neq B$, and CE meets ω at $P \neq C$. The bisector of $\angle PMC$ meets segment AC at $Q \neq C$. Given that $PQ = MC$, determine the degree measure of $\angle ABC$.

Ray Li.

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28 Find the remainder when

$$\sum_{k=1}^{2^{16}} \binom{2k}{k}(3 \cdot 2^{14}+1)^k (k-1)^{2^{16}-1}$$

is divided by $2^{16}+1$. (*Note:* It is well-known that $2^{16}+1 = 65537$ is prime.)

Victor Wang.

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29 In the Cartesian plane, let $S_{i,j} = \{(x,y) \mid i \le x \le j\}$. For $i = 0, 1, \ldots, 2012$, color $S_{i,i+1}$ pink if i is even and gray if i is odd. For a convex polygon P in the plane, let $d(P)$ denote its pink density, i.e. the fraction of its total area that is pink. Call a polygon P *pinxtreme* if it lies completely in the region $S_{0,2013}$ and has at least one vertex on each of the lines $x = 0$ and $x = 2013$. Given that the minimum value of $d(P)$ over all non-degenerate convex pinxtreme polygons P in the plane can be expressed in the form $\frac{(1+\sqrt{p})^2}{q^2}$ for positive integers p, q, find $p+q$.

Victor Wang.

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30 Let $P(x)$ denote the polynomial

$$3\sum_{k=0}^{9} x^k + 2\sum_{k=10}^{1209} x^k + \sum_{k=1210}^{146409} x^k.$$

Find the smallest positive integer n for which there exist polynomials f, g with integer coefficients satisfying $x^n - 1 = (x^{16}+1)P(x)f(x) + 11 \cdot g(x)$.

Victor Wang.
