Online Math Open Problems 2013

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by v_Enhance, math154

–	Winter

1   Let x be the answer to this problem. For what real number a is the answer to this problem also $a - x$?

Ray Li

2   The number 123454321 is written on a blackboard. Evan walks by and erases some (but not all) of the digits, and notices that the resulting number (when spaces are removed) is divisible by 9. What is the fewest number of digits he could have erased?

Ray Li

3   Three lines m, n, and ℓ lie in a plane such that no two are parallel. Lines m and n meet at an acute angle of $14°$, and lines m and ℓ meet at an acute angle of $20°$. Find, in degrees, the sum of all possible acute angles formed by lines n and ℓ.

Ray Li

4   For how many ordered pairs of positive integers (a, b) with $a, b < 1000$ is it true that a times b is equal to b^2 divided by a? For example, 3 times 9 is equal to 9^2 divided by 3.

Ray Li

5   At the Mountain School, Micchell is assigned a *submissiveness rating* of 3.0 or 4.0 for each class he takes. His *college potential* is then defined as the average of his submissiveness ratings over all classes taken. After taking 40 classes, Micchell has a college potential of 3.975. Unfortunately, he needs a college potential of at least 3.995 to get into the South Harmon Institute of Technology (`http://en.wikipedia.org/wiki/Accepte\#Plot`). Otherwise, he becomes a rock. Assuming he receives a submissiveness rating of 4.0 in every class he takes from now on, how many more classes does he need to take in order to get into the South Harmon Institute of Technology?

Victor Wang

6   Circle S_1 has radius 5. Circle S_2 has radius 7 and has its center lying on S_1. Circle S_3 has an integer radius and has its center lying on S_2. If the center of S_1 lies on S_3, how many possible values are there for the radius of S_3?

Ray Li

7   Jacob's analog clock has 12 equally spaced tick marks on the perimeter, but all the digits have been erased, so he doesn't know which tick mark corresponds to which hour. Jacob takes an arbitrary tick mark and measures clockwise to the hour hand and minute hand. He measures that the minute hand is 300 degrees clockwise of the tick mark, and that the hour hand is 70 degrees clockwise of the same tick mark. If it is currently morning, how many minutes past midnight is it?

Ray Li

8   How many ways are there to choose (not necessarily distinct) integers a, b, c from the set $\{1, 2, 3, 4\}$ such that $a^{(b^c)}$ is divisible by 4?

Ray Li

9   David has a collection of 40 rocks, 30 stones, 20 minerals and 10 gemstones. An operation consists of removing three objects, no two of the same type. What is the maximum number of operations he can possibly perform?

Ray Li

10   At certain store, a package of 3 apples and 12 oranges costs 5 dollars, and a package of 20 apples and 5 oranges costs 13 dollars. Given that apples and oranges can only be bought in these two packages, what is the minimum nonzero amount of dollars that must be spent to have an equal number of apples and oranges?

Ray Li

11   Let A, B, and C be distinct points on a line with $AB = AC = 1$. Square $ABDE$ and equilateral triangle ACF are drawn on the same side of line BC. What is the degree measure of the acute angle formed by lines EC and BF?

Ray Li

12   There are 25 ants on a number line; five at each of the coordinates $1, 2, 3, 4,$ and 5. Each minute, one ant moves from its current position to a position one unit away. What is the minimum number of minutes that must pass before it is possible for no two ants to be on the same coordinate?

Ray Li

13   There are three flies of negligible size that start at the same position on a circular track with circumference 1000 meters. They fly clockwise at speeds of 2, 6, and k meters per second, respectively, where k is some positive integer with $7 \leq k \leq 2013$. Suppose that at some point

in time, all three flies meet at a location different from their starting point. How many possible values of k are there?

Ray Li

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14 What is the smallest perfect square larger than 1 with a perfect square number of positive integer factors?

Ray Li

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15 A permutation $a_1, a_2, ..., a_{13}$ of the numbers from 1 to 13 is given such that $a_i > 5$ for $i = 1, 2, 3, 4, 5$. Determine the maximum possible value of

$$a_{a_1} + a_{a_2} + a_{a_3} + a_{a_4} + a_{a_5}.$$

Evan Chen

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16 Let S_1 and S_2 be two circles intersecting at points A and B. Let C and D be points on S_1 and S_2 respectively such that line CD is tangent to both circles and A is closer to line CD than B. If $\angle BCA = 52°$ and $\angle BDA = 32°$, determine the degree measure of $\angle CBD$.

Ray Li

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17 Determine the number of ordered pairs of positive integers (x, y) with $y < x \leq 100$ such that $x^2 - y^2$ and $x^3 - y^3$ are relatively prime. (Two numbers are *relatively prime* if they have no common factor other than 1.)

Ray Li

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18 Determine the absolute value of the sum

$$\lfloor 2013 \sin 0° \rfloor + \lfloor 2013 \sin 1° \rfloor + \cdots + \lfloor 2013 \sin 359° \rfloor,$$

where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

(You may use the fact that $\sin n°$ is irrational for positive integers n not divisible by 30.)

Ray Li

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19 A, B, C are points in the plane such that $\angle ABC = 90°$. Circles with diameters BA and BC meet at D. If $BA = 20$ and $BC = 21$, then the length of segment BD can be expressed in the form $\frac{m}{n}$ where m and n are relatively prime positive integers. What is $m + n$?

Ray Li

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20 Let $a_1, a_2, \ldots, a_{2013}$ be a permutation of the numbers from 1 to 2013. Let $A_n = \frac{a_1 + a_2 + \cdots + a_n}{n}$ for $n = 1, 2, \ldots, 2013$. If the smallest possible difference between the largest and smallest values of $A_1, A_2, \ldots, A_{2013}$ is $\frac{m}{n}$, where m and n are relatively prime positive integers, find $m + n$.

Ray Li

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21   Dirock has a very neat rectangular backyard that can be represented as a 32×32 grid of unit squares. The rows and columns are each numbered $1, 2, \ldots, 32$. Dirock is very fond of rocks, and places a rock in every grid square whose row and column number are both divisible by 3. Dirock would like to build a rectangular fence with vertices at the centers of grid squares and sides parallel to the sides of the yard such that
- The fence does not pass through any grid squares containing rocks; - The interior of the fence contains exactly 5 rocks.
In how many ways can this be done?

Ray Li

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22   In triangle ABC, $AB = 28$, $AC = 36$, and $BC = 32$. Let D be the point on segment BC satisfying $\angle BAD = \angle DAC$, and let E be the unique point such that $DE \parallel AB$ and line AE is tangent to the circumcircle of ABC. Find the length of segment AE.

Ray Li

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23   A set of 10 distinct integers S is chosen. Let M be the number of nonempty subsets of S whose elements have an even sum. What is the minimum possible value of M?

- S is the "set of 10 distinct integers" from the first sentence.

Ray Li

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24   For a permutation π of the integers from 1 to 10, define

$$S(\pi) = \sum_{i=1}^{9} (\pi(i) - \pi(i+1)) \cdot (4 + \pi(i) + \pi(i+1)),$$

where $\pi(i)$ denotes the ith element of the permutation. Suppose that M is the maximum possible value of $S(\pi)$ over all permutations π of the integers from 1 to 10. Determine the number of permutations π for which $S(\pi) = M$.

Ray Li

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25   Positive integers $x, y, z \le 100$ satisfy

$$1099x + 901y + 1110z = 59800$$
$$109x + 991y + 101z = 44556$$

Compute $10000x + 100y + z$.

Evan Chen

26　　In triangle ABC, F is on segment AB such that CF bisects $\angle ACB$. Points D and E are on line CF such that lines AD, BE are perpendicular to CF. M is the midpoint of AB. If $ME = 13$, $AD = 15$, and $BE = 25$, find $AC + CB$.

Ray Li

27　　Geodude wants to assign one of the integers $1, 2, 3, \ldots, 11$ to each lattice point (x, y, z) in a 3D Cartesian coordinate system. In how many ways can Geodude do this if for every lattice parallelogram $ABCD$, the positive difference between the sum of the numbers assigned to A and C and the sum of the numbers assigned to B and D must be a multiple of 11? (A *lattice point* is a point with all integer coordinates. A *lattice parallelogram* is a parallelogram with all four vertices lying on lattice points. Here, we say four not necessarily distinct points A, B, C, D form a *parallelogram* $ABCD$ if and only if the midpoint of segment AC coincides with the midpoint of segment BD.)

- The "positive difference" between two real numbers x and y is the quantity $|x - y|$. Note that this may be zero.
- The last sentence was added to remove confusion about "degenerate parallelograms."

Victor Wang

28　　Let S be the set of all lattice points (x, y) in the plane satisfying $|x| + |y| \le 10$. Let $P_1, P_2, \ldots, P_{2013}$ be a sequence of 2013 (not necessarily distinct) points such that for every point Q in S, there exists at least one index i such that $1 \le i \le 2013$ and $P_i = Q$. Suppose that the minimum possible value of $|P_1P_2| + |P_2P_3| + \cdots + |P_{2012}P_{2013}|$ can be expressed in the form $a + b\sqrt{c}$, where a, b, c are positive integers and c is not divisible by the square of any prime. Find $a + b + c$. (A *lattice point* is a point with all integer coordinates.)

- $k = 2013$, i.e. the problem should read, "... there exists at least one index i such that $1 \le i \le 2013$ ...". An earlier version of the test read $1 \le i \le k$.

Anderson Wang

29　　Let $\phi(n)$ denote the number of positive integers less than or equal to n that are relatively prime to n, and let $d(n)$ denote the number of positive integer divisors of n. For example, $\phi(6) = 2$ and $d(6) = 4$. Find the sum of all odd integers $n \le 5000$ such that $n \mid \phi(n)d(n)$.

Alex Zhu

30　　Pairwise distinct points $P_1, P_2, \ldots, P_{16}$ lie on the perimeter of a square with side length 4 centered at O such that $|P_iP_{i+1}| = 1$ for $i = 1, 2, \ldots, 16$. (We take P_{17} to be the point P_1.) We construct points $Q_1, Q_2, \ldots, Q_{16}$ as follows: for each i, a fair coin is flipped. If it lands heads, we define Q_i to be P_i; otherwise, we define Q_i to be the reflection of P_i over O. (So, it is possible

for some of the Q_i to coincide.) Let D be the length of the vector $\overrightarrow{OQ_1} + \overrightarrow{OQ_2} + \cdots + \overrightarrow{OQ_{16}}$. Compute the expected value of D^2.

Ray Li

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31   Beyond the Point of No Return is a large lake containing 2013 islands arranged at the vertices of a regular 2013-gon. Adjacent islands are joined with exactly two bridges. Christine starts on one of the islands with the intention of burning all the bridges. Each minute, if the island she is on has at least one bridge still joined to it, she randomly selects one such bridge, crosses it, and immediately burns it. Otherwise, she stops.

If the probability Christine burns all the bridges before she stops can be written as $\frac{m}{n}$ for relatively prime positive integers m and n, find the remainder when $m + n$ is divided by 1000.

Evan Chen

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32   In $\triangle ABC$ with incenter I, $AB = 61$, $AC = 51$, and $BC = 71$. The circumcircles of triangles AIB and AIC meet line BC at points D $(D \neq B)$ and E $(E \neq C)$, respectively. Determine the length of segment DE.

James Tao

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33   Let n be a positive integer. E. Chen and E. Chen play a game on the n^2 points of an $n \times n$ lattice grid. They alternately mark points on the grid such that no player marks a point that is on or inside a non-degenerate triangle formed by three marked points. Each point can be marked only once. The game ends when no player can make a move, and the last player to make a move wins. Determine the number of values of n between 1 and 2013 (inclusive) for which the first player can guarantee a win, regardless of the moves that the second player makes.

Ray Li

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34   For positive integers n, let $s(n)$ denote the sum of the squares of the positive integers less than or equal to n that are relatively prime to n. Find the greatest integer less than or equal to

$$\sum_{n|2013} \frac{s(n)}{n^2},$$

where the summation runs over all positive integers n dividing 2013.

Ray Li

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35   The rows and columns of a 7×7 grid are each numbered $1, 2, \ldots, 7$. In how many ways can one choose 8 cells of this grid such that for every two chosen cells X and Y, either the positive difference of their row numbers is at least 3, or the positive difference of their column numbers is at least 3?

- The "or" here is inclusive (as by convention, despite the "either"), i.e. X and Y are permitted if and only if they satisfy the row condition, the column condition, or both.

Ray Li

36   Let $ABCD$ be a nondegenerate isosceles trapezoid with integer side lengths such that $BC \parallel AD$ and $AB = BC = CD$. Given that the distance between the incenters of triangles ABD and ACD is $8!$, determine the number of possible lengths of segment AD.

Ray Li

37   Let M be a positive integer. At a party with 120 people, 30 wear red hats, 40 wear blue hats, and 50 wear green hats. Before the party begins, M pairs of people are friends. (Friendship is mutual.) Suppose also that no two friends wear the same colored hat to the party.

During the party, X and Y can become friends if and only if the following two conditions hold:
- There exists a person Z such that X and Y are both friends with Z. (The friendship(s) between Z, X and Z, Y could have been formed during the party.) - X and Y are not wearing the same colored hat.

Suppose the party lasts long enough so that all possible friendships are formed. Let M_1 be the largest value of M such that regardless of which M pairs of people are friends before the party, there will always be at least one pair of people X and Y with different colored hats who are not friends after the party. Let M_2 be the smallest value of M such that regardless of which M pairs of people are friends before the party, every pair of people X and Y with different colored hats are friends after the party. Find $M_1 + M_2$.

- The definition of M_2 should read, "Let M_2 be the *smallest* value of M such that...". An earlier version of the test read "largest value of M".

Victor Wang

38   Triangle ABC has sides $AB = 25$, $BC = 30$, and $CA = 20$. Let P, Q be the points on segments AB, AC, respectively, such that $AP = 5$ and $AQ = 4$. Suppose lines BQ and CP intersect at R and the circumcircles of $\triangle BPR$ and $\triangle CQR$ intersect at a second point $S \neq R$. If the length of segment SA can be expressed in the form $\frac{m}{\sqrt{n}}$ for positive integers m, n, where n is not divisible by the square of any prime, find $m + n$.

Victor Wang

39   Find the number of 8-digit base-6 positive integers $(a_1a_2a_3a_4a_5a_6a_7a_8)_6$ (with leading zeros permitted) such that $(a_1a_2 \ldots a_8)_6 \mid (a_{i+1}a_{i+2} \ldots a_{i+8})_6$ for $i = 1, 2, \ldots, 7$, where indices are taken modulo 8 (so $a_9 = a_1$, $a_{10} = a_2$, and so on).

Victor Wang

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40 Let ABC be a triangle with $AB = 13$, $BC = 14$, and $AC = 15$. Let M be the midpoint of BC and let Γ be the circle passing through A and tangent to line BC at M. Let Γ intersect lines AB and AC at points D and E, respectively, and let N be the midpoint of DE. Suppose line MN intersects lines AB and AC at points P and O, respectively. If the ratio $MN : NO : OP$ can be written in the form $a : b : c$ with a, b, c positive integers satisfying $\gcd(a, b, c) = 1$, find $a + b + c$.

James Tao

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41 While there do not exist pairwise distinct real numbers a, b, c satisfying $a^2 + b^2 + c^2 = ab + bc + ca$, there do exist complex numbers with that property. Let a, b, c be complex numbers such that $a^2 + b^2 + c^2 = ab + bc + ca$ and $|a + b + c| = 21$. Given that $|a - b| = 2\sqrt{3}$, $|a| = 3\sqrt{3}$, compute $|b|^2 + |c|^2$.

- The problem should read $|a + b + c| = 21$. An earlier version of the test read $|a + b + c| = 7$; that value is incorrect.
- $|b|^2 + |c|^2$ should be a positive integer, not a fraction; an earlier version of the test read "... for relatively prime positive integers m and n. Find $m + n$."

Ray Li

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42 Find the remainder when

$$\prod_{i=0}^{100} (1 - i^2 + i^4)$$

is divided by 101.

Victor Wang

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43 In a tennis tournament, each competitor plays against every other competitor, and there are no draws. Call a group of four tennis players "ordered" if there is a clear winner and a clear loser (i.e., one person who beat the other three, and one person who lost to the other three.) Find the smallest integer n for which any tennis tournament with n people has a group of four tennis players that is ordered.

Ray Li

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44 Suppose tetrahedron $PABC$ has volume 420 and satisfies $AB = 13$, $BC = 14$, and $CA = 15$. The minimum possible surface area of $PABC$ can be written as $m + n\sqrt{k}$, where m, n, k are positive integers and k is not divisible by the square of any prime. Compute $m + n + k$.

Ray Li

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45   Let N denote the number of ordered 2011-tuples of positive integers $(a_1, a_2, \ldots, a_{2011})$ with $1 \le a_1, a_2, \ldots, a_{2011} \le 2011^2$ such that there exists a polynomial f of degree 4019 satisfying the following three properties:
- $f(n)$ is an integer for every integer n; - $2011^2 \mid f(i) - a_i$ for $i = 1, 2, \ldots, 2011$; - $2011^2 \mid f(n + 2011) - f(n)$ for every integer n.
Find the remainder when N is divided by 1000.

Victor Wang

46   Let ABC be a triangle with $\angle B - \angle C = 30°$. Let D be the point where the A-excircle touches line BC, O the circumcenter of triangle ABC, and X, Y the intersections of the altitude from A with the incircle with X in between A and Y. Suppose points A, O and D are collinear. If the ratio $\frac{AO}{AX}$ can be expressed in the form $\frac{a + b\sqrt{c}}{d}$ for positive integers a, b, c, d with $\gcd(a, b, d) = 1$ and c not divisible by the square of any prime, find $a + b + c + d$.

James Tao

47   Let $f(x, y)$ be a function from ordered pairs of positive integers to real numbers such that

$$f(1, x) = f(x, 1) = \frac{1}{x} \quad \text{and} \quad f(x + 1, y + 1)f(x, y) - f(x, y + 1)f(x + 1, y) = 1$$

for all ordered pairs of positive integers (x, y). If $f(100, 100) = \frac{m}{n}$ for two relatively prime positive integers m, n, compute $m + n$.

David Yang

48 ω is a complex number such that $\omega^{2013} = 1$ and $\omega^m \neq 1$ for $m = 1, 2, \ldots, 2012$. Find the number of ordered pairs of integers (a, b) with $1 \le a, b \le 2013$ such that

$$\frac{(1 + \omega + \cdots + \omega^a)(1 + \omega + \cdots + \omega^b)}{3}$$

is the root of some polynomial with integer coefficients and leading coefficient 1. (Such complex numbers are called *algebraic integers*.)

Victor Wang

49   In $\triangle ABC$, $CA = 1960\sqrt{2}$, $CB = 6720$, and $\angle C = 45°$. Let K, L, M lie on BC, CA, and AB such that $AK \perp BC$, $BL \perp CA$, and $AM = BM$. Let N, O, P lie on KL, BA, and BL such that $AN = KN$, $BO = CO$, and A lies on line NP. If H is the orthocenter of $\triangle MOP$, compute HK^2.

- Without further qualification, "XY" denotes line XY.

Evan Chen

50 Let S denote the set of words $W = w_1 w_2 \ldots w_n$ of any length $n \geq 0$ (including the empty string λ), with each letter w_i from the set $\{x, y, z\}$. Call two words U, V *similar* if we can insert a string $s \in \{xyz, yzx, zxy\}$ of three consecutive letters somewhere in U (possibly at one of the ends) to obtain V or somewhere in V (again, possibly at one of the ends) to obtain U, and say a word W is *trivial* if for some nonnegative integer m, there exists a sequence $W_0, W_1, \ldots, W_m$ such that $W_0 = \lambda$ is the empty string, $W_m = W$, and W_i, W_{i+1} are similar for $i = 0, 1, \ldots, m - 1$. Given that for two relatively prime positive integers p, q we have

$$\frac{p}{q} = \sum_{n \geq 0} f(n) \left(\frac{225}{8192} \right)^n,$$

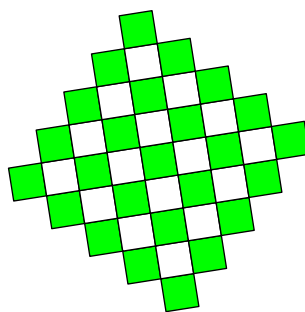where $f(n)$ denotes the number of trivial words in S of length $3n$ (in particular, $f(0) = 1$), find $p + q$.

Victor Wang

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1 Determine the value of $142857 + 285714 + 428571 + 571428$.

Proposed by Ray Li

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2 The figure below consists of several unit squares, M of which are white and N of which are green. Compute $100M + N$.

Proposed by Evan Chen

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3 A *palindromic table* is a 3×3 array of letters such that the words in each row and column read the same forwards and backwards. An example of such a table is shown below.

$$\begin{array}{ccc} O & M & O \\ N & M & N \\ O & M & O \end{array}$$

How many palindromic tables are there that use only the letters O and M? (The table may contain only a single letter.)

Proposed by Evan Chen

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4   Suppose $a_1, a_2, a_3, \ldots$ is an increasing arithmetic progression of positive integers. Given that $a_3 = 13$, compute the maximum possible value of

$$a_{a_1} + a_{a_2} + a_{a_3} + a_{a_4} + a_{a_5}.$$

Proposed by Evan Chen

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5   A wishing well is located at the point $(11, 11)$ in the xy-plane. Rachelle randomly selects an integer y from the set $\{0, 1, \ldots, 10\}$. Then she randomly selects, with replacement, two integers a, b from the set $\{1, 2, \ldots, 10\}$. The probability the line through $(0, y)$ and (a, b) passes through the well can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute $m + n$.

Proposed by Evan Chen

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6   Find the number of integers n with $n \geq 2$ such that the remainder when 2013 is divided by n is equal to the remainder when n is divided by 3.

Proposed by Michael Kural

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7   Points M, N, P are selected on sides $\overline{AB}, \overline{AC}, \overline{BC}$, respectively, of triangle ABC. Find the area of triangle MNP given that $AM = MB = BP = 15$ and $AN = NC = CP = 25$.

Proposed by Evan Chen

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8   Suppose that $x_1 < x_2 < \cdots < x_n$ is a sequence of positive integers such that x_k divides x_{k+2} for each $k = 1, 2, \ldots, n - 2$. Given that $x_n = 1000$, what is the largest possible value of n?

Proposed by Evan Chen

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9   Let $AXYZB$ be a regular pentagon with area 5 inscribed in a circle with center O. Let Y' denote the reflection of Y over $\overline{AB}$ and suppose C is the center of a circle passing through A, Y' and B. Compute the area of triangle ABC.

Proposed by Evan Chen

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10   In convex quadrilateral $AEBC$, $\angle BEA = \angle CAE = 90°$ and $AB = 15$, $BC = 14$ and $CA = 13$. Let D be the foot of the altitude from C to $\overline{AB}$. If ray CD meets $\overline{AE}$ at F, compute $AE \cdot AF$.

Proposed by David Stoner

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11   Four orange lights are located at the points $(2,0)$, $(4,0)$, $(6,0)$ and $(8,0)$ in the xy-plane. Four yellow lights are located at the points $(1,0)$, $(3,0)$, $(5,0)$, $(7,0)$. Sparky chooses one or more of the lights to turn on. In how many ways can he do this such that the collection of illuminated lights is symmetric around some line parallel to the y-axis?

Proposed by Evan Chen

12   Let a_n denote the remainder when $(n+1)^3$ is divided by n^3; in particular, $a_1 = 0$. Compute the remainder when $a_1 + a_2 + \cdots + a_{2013}$ is divided by 1000.

Proposed by Evan Chen

13   In the rectangular table shown below, the number 1 is written in the upper-left hand corner, and every number is the sum of the any numbers directly to its left and above. The table extends infinitely downwards and to the right.

$$
\begin{array}{cccccc}
1 & 1 & 1 & 1 & 1 & \cdots \\
1 & 2 & 3 & 4 & 5 & \cdots \\
1 & 3 & 6 & 10 & 15 & \cdots \\
1 & 4 & 10 & 20 & 35 & \cdots \\
1 & 5 & 15 & 35 & 70 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}
$$

Wanda the Worm, who is on a diet after a feast two years ago, wants to eat n numbers (not necessarily distinct in value) from the table such that the sum of the numbers is less than one million. However, she cannot eat two numbers in the same row or column (or both). What is the largest possible value of n?

Proposed by Evan Chen

14   In the universe of Pi Zone, points are labeled with 2×2 arrays of positive reals. One can teleport from point M to point M' if M can be obtained from M' by multiplying either a row or column by some positive real. For example, one can teleport from $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ to $\begin{pmatrix} 1 & 20 \\ 3 & 40 \end{pmatrix}$ and then to $\begin{pmatrix} 1 & 20 \\ 6 & 80 \end{pmatrix}$.

A *tourist attraction* is a point where each of the entries of the associated array is either 1, 2, 4, 8 or 16. A company wishes to build a hotel on each of several points so that at least one hotel is accessible from every tourist attraction by teleporting, possibly multiple times. What is the minimum number of hotels necessary?

Proposed by Michael Kural

15   Find the positive integer n such that

$$\underbrace{f(f(\cdots f(n)\cdots))}_{2013\ f\text{'s}} = 2014^2 + 1$$

where $f(n)$ denotes the nth positive integer which is not a perfect square.

Proposed by David Stoner

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16   Al has the cards $1, 2, \ldots, 10$ in a row in increasing order. He first chooses the cards labeled 1, 2, and 3, and rearranges them among their positions in the row in one of six ways (he can leave the positions unchanged). He then chooses the cards labeled 2, 3, and 4, and rearranges them among their positions in the row in one of six ways. (For example, his first move could have made the sequence $3, 2, 1, 4, 5, \ldots$, and his second move could have rearranged that to $2, 4, 1, 3, 5, \ldots$.) He continues this process until he has rearranged the cards with labels $8, 9, 10$. Determine the number of possible orderings of cards he can end up with.

Proposed by Ray Li

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17   Let $ABXC$ be a parallelogram. Points K, P, Q lie on $\overline{BC}$ in this order such that $BK = \frac{1}{3}KC$ and $BP = PQ = QC = \frac{1}{3}BC$. Rays XP and XQ meet $\overline{AB}$ and $\overline{AC}$ at D and E, respectively. Suppose that $\overline{AK} \perp \overline{BC}$, $EK - DK = 9$ and $BC = 60$. Find $AB + AC$.

Proposed by Evan Chen

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18   Given an $n \times n$ grid of dots, let $f(n)$ be the largest number of segments between adjacent dots which can be drawn such that (i) at most one segment is drawn between each pair of dots, and (ii) each dot has 1 or 3 segments coming from it. (For example, $f(4) = 16$.) Compute $f(2000)$.

Proposed by David Stoner

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19   Let $\sigma(n)$ be the number of positive divisors of n, and let $\operatorname{rad} n$ be the product of the distinct prime divisors of n. By convention, $\operatorname{rad} 1 = 1$. Find the greatest integer not exceeding

$$100 \left(\sum_{n=1}^{\infty} \frac{\sigma(n)\sigma(n\operatorname{rad} n)}{n^2\sigma(\operatorname{rad} n)} \right)^{\frac{1}{3}}.$$

Proposed by Michael Kural

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20   A positive integer n is called *mythical* if every divisor of n is two less than a prime. Find the unique mythical number with the largest number of divisors.

Proposed by Evan Chen

21   Let ABC be a triangle with $AB = 5$, $AC = 8$, and $BC = 7$. Let D be on side AC such that $AD = 5$ and $CD = 3$. Let I be the incenter of triangle ABC and E be the intersection of the perpendicular bisectors of $\overline{ID}$ and $\overline{BC}$. Suppose $DE = \frac{a\sqrt{b}}{c}$ where a and c are relatively prime positive integers, and b is a positive integer not divisible by the square of any prime. Find $a + b + c$.

Proposed by Ray Li

22   Find the sum of all integers m with $1 \le m \le 300$ such that for any integer n with $n \ge 2$, if $2013m$ divides $n^n - 1$ then $2013m$ also divides $n - 1$.

Proposed by Evan Chen

23   Let $ABCDE$ be a regular pentagon, and let F be a point on $\overline{AB}$ with $\angle CDF = 55°$. Suppose $\overline{FC}$ and $\overline{BE}$ meet at G, and select H on the extension of $\overline{CE}$ past E such that $\angle DHE = \angle FDG$. Find the measure of $\angle GHD$, in degrees.

Proposed by David Stoner

24   The real numbers $a_0, a_1, \ldots, a_{2013}$ and $b_0, b_1, \ldots, b_{2013}$ satisfy $a_n = \frac{1}{63}\sqrt{2n+2} + a_{n-1}$ and $b_n = \frac{1}{96}\sqrt{2n+2} - b_{n-1}$ for every integer $n = 1, 2, \ldots, 2013$. If $a_0 = b_{2013}$ and $b_0 = a_{2013}$, compute

$$\sum_{k=1}^{2013} \left(a_k b_{k-1} - a_{k-1} b_k \right).$$

Proposed by Evan Chen

25   Let $ABCD$ be a quadrilateral with $AD = 20$ and $BC = 13$. The area of $\triangle ABC$ is 338 and the area of $\triangle DBC$ is 212. Compute the smallest possible perimeter of $ABCD$.

Proposed by Evan Chen

26   Let ABC be a triangle with $AB = 13$, $AC = 25$, and $\tan A = \frac{3}{4}$. Denote the reflections of B, C across $\overline{AC}, \overline{AB}$ by D, E, respectively, and let O be the circumcenter of triangle ABC. Let P be a point such that $\triangle DPO \sim \triangle PEO$, and let X and Y be the midpoints of the major and minor arcs $\overset{\frown}{BC}$ of the circumcircle of triangle ABC. Find $PX \cdot PY$.

Proposed by Michael Kural

27   Ben has a big blackboard, initially empty, and Francisco has a fair coin. Francisco flips the coin 2013 times. On the nth flip (where $n = 1, 2, \ldots, 2013$), Ben does the following if the coin flips heads:
(i) If the blackboard is empty, Ben writes n on the blackboard.
(ii) If the blackboard is not empty, let m denote the largest number on the blackboard. If $m^2 + 2n^2$ is divisible by 3, Ben erases m from the blackboard; otherwise, he writes the number n.

No action is taken when the coin flips tails. If probability that the blackboard is empty after all 2013 flips is $\frac{2u+1}{2^k(2v+1)}$, where u, v, and k are nonnegative integers, compute k.

Proposed by Evan Chen

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28   Let n denote the product of the first 2013 primes. Find the sum of all primes p with $20 \le p \le 150$ such that
(i) $\frac{p+1}{2}$ is even but is not a power of 2, and
(ii) there exist pairwise distinct positive integers a, b, c for which

$$a^n(a - b)(a - c) + b^n(b - c)(b - a) + c^n(c - a)(c - b)$$

is divisible by p but not p^2.

Proposed by Evan Chen

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29   Kevin has 255 cookies, each labeled with a unique nonempty subset of $\{1, 2, 3, 4, 5, 6, 7, 8\}$. Each day, he chooses one cookie uniformly at random out of the cookies not yet eaten. Then, he eats that cookie, and all remaining cookies that are labeled with a subset of that cookie (for example, if he chooses the cookie labeled with $\{1, 2\}$, he eats that cookie as well as the cookies with $\{1\}$ and $\{2\}$). The expected value of the number of days that Kevin eats a cookie before all cookies are gone can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Proposed by Ray Li

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30   Let $P(t) = t^3 + 27t^2 + 199t + 432$. Suppose a, b, c, and x are distinct positive reals such that $P(-a) = P(-b) = P(-c) = 0$, and

$$\sqrt{\frac{a + b + c}{x}} = \sqrt{\frac{b + c + x}{a}} + \sqrt{\frac{c + a + x}{b}} + \sqrt{\frac{a + b + x}{c}}.$$

If $x = \frac{m}{n}$ for relatively prime positive integers m and n, compute $m + n$.

Proposed by Evan Chen