Online Math Open Problems 2014

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by v_Enhance

– Spring

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1 In English class, you have discovered a mysterious phenomenon – if you spend n hours on an essay, your score on the essay will be $100\left(1-4^{-n}\right)$ points if $2n$ is an integer, and 0 otherwise. For example, if you spend 30 minutes on an essay you will get a score of 50, but if you spend 35 minutes on the essay you somehow do not earn any points.

It is 4AM, your English class starts at 8:05AM the same day, and you have four essays due at the start of class. If you can only work on one essay at a time, what is the maximum possible average of your essay scores?

Proposed by Evan Chen

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2 Consider two circles of radius one, and let O and O' denote their centers. Point M is selected on either circle. If $OO' = 2014$, what is the largest possible area of triangle OMO'?

Proposed by Evan Chen

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3 Suppose that m and n are relatively prime positive integers with $A = \frac{m}{n}$, where

$$A = \frac{2+4+6+\dots+2014}{1+3+5+\dots+2013} - \frac{1+3+5+\dots+2013}{2+4+6+\dots+2014}.$$

Find m. In other words, find the numerator of A when A is written as a fraction in simplest form.

Proposed by Evan Chen

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4 The integers $1, 2, \dots, n$ are written in order on a long slip of paper. The slip is then cut into five pieces, so that each piece consists of some (nonempty) consecutive set of integers. The averages of the numbers on the five slips are 1234, 345, 128, 19, and 9.5 in some order. Compute n.

Proposed by Evan Chen

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5 Joe the teacher is bad at rounding. Because of this, he has come up with his own way to round grades, where a *grade* is a nonnegative decimal number with finitely many digits after the decimal point.

Given a grade with digits $a_1a_2\dots a_m.b_1b_2\dots b_n$, Joe first rounds the number to the nearest 10^{-n+1}th place. He then repeats the procedure on the new number, rounding to the nearest

10^{-n+2}th, then rounding the result to the nearest 10^{-n+3}th, and so on, until he obtains an integer. For example, he rounds the number 2014.456 via $2014.456 \to 2014.46 \to 2014.5 \to 2015$.

There exists a rational number M such that a grade x gets rounded to at least 90 if and only if $x \geq M$. If $M = \frac{p}{q}$ for relatively prime integers p and q, compute $p+q$.

Proposed by Yang Liu

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6 Let L_n be the least common multiple of the integers $1, 2, \dots, n$. For example, $L_{10} = 2{,}520$ and $L_{30} = 2{,}329{,}089{,}562{,}800$. Find the remainder when L_{31} is divided by $100{,}000$.

Proposed by Evan Chen

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7 How many integers n with $10 \le n \le 500$ have the property that the hundreds digit of $17n$ and $17n+17$ are different?

Proposed by Evan Chen

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8 Let a_1, a_2, a_3, a_4, a_5 be real numbers satisfying

$$\begin{aligned} 2a_1+a_2+a_3+a_4+a_5 &= 1 + \tfrac{1}{8}a_4 \\ 2a_2+a_3+a_4+a_5 &= 2 + \tfrac{1}{4}a_3 \\ 2a_3+a_4+a_5 &= 4 + \tfrac{1}{2}a_2 \\ 2a_4+a_5 &= 6 + a_1 \end{aligned}$$

Compute $a_1+a_2+a_3+a_4+a_5$.

Proposed by Evan Chen

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9 Eighteen students participate in a team selection test with three problems, each worth up to seven points. All scores are nonnegative integers. After the competition, the results are posted by Evan in a table with 3 columns: the student's name, score, and rank (allowing ties), respectively. Here, a student's rank is one greater than the number of students with strictly higher scores (for example, if seven students score $0, 0, 7, 8, 8, 14, 21$ then their ranks would be $6, 6, 5, 3, 3, 2, 1$ respectively).

When Richard comes by to read the results, he accidentally reads the rank column as the score column and vice versa. Coincidentally, the results still made sense! If the scores of the students were $x_1 \le x_2 \le \dots \le x_{18}$, determine the number of possible values of the 18-tuple $(x_1, x_2, \dots, x_{18})$. In other words, determine the number of possible multisets (sets with repetition) of scores.

Proposed by Yang Liu

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10 Let $A_1A_2\dots A_{4000}$ be a regular 4000-gon. Let X be the foot of the altitude from A_{1986} onto diagonal $A_{1000}A_{3000}$, and let Y be the foot of the altitude from A_{2014} onto $A_{2000}A_{4000}$. If $XY=1$, what is the area of square $A_{500}A_{1500}A_{2500}A_{3500}$?

Proposed by Evan Chen

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11 Let X be a point inside convex quadrilateral $ABCD$ with $\angle AXB+\angle CXD=180^\circ$. If $AX=14$, $BX=11$, $CX=5$, $DX=10$, and $AB=CD$, find the sum of the areas of $\triangle AXB$ and $\triangle CXD$.

Proposed by Michael Kural

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12 The points A, B, C, D, E lie on a line ℓ in this order. Suppose T is a point not on ℓ such that $\angle BTC=\angle DTE$, and $\overline{AT}$ is tangent to the circumcircle of triangle BTE. If $AB=2$, $BC=36$, and $CD=15$, compute DE.

Proposed by Yang Liu

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13 Suppose that g and h are polynomials of degree 10 with integer coefficients such that $g(2)<h(2)$ and

$$g(x)h(x)=\sum_{k=0}^{10}\left(\binom{k+11}{k}x^{20-k}-\binom{21-k}{11}x^{k-1}+\binom{21}{11}x^{k-1}\right)$$

holds for all nonzero real numbers x. Find $g(2)$.

Proposed by Yang Liu

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14 Let ABC be a triangle with incenter I and $AB=1400$, $AC=1800$, $BC=2014$. The circle centered at I passing through A intersects line BC at two points X and Y. Compute the length XY.

Proposed by Evan Chen

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15 In Prime Land, there are seven major cities, labelled C_0, C_1, …, C_6. For convenience, we let $C_{n+7}=C_n$ for each $n=0,1,\dots,6$; i.e. we take the indices modulo 7. Al initially starts at city C_0.

Each minute for ten minutes, Al flips a fair coin. If the coin land heads, and he is at city C_k, he moves to city C_{2k}; otherwise he moves to city C_{2k+1}. If the probability that Al is back at city C_0 after 10 moves is $\frac{m}{1024}$, find m.

Proposed by Ray Li

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16 Say a positive integer n is *radioactive* if one of its prime factors is strictly greater than $\sqrt{n}$. For example, $2012=2^2\cdot 503$, $2013=3\cdot 11\cdot 61$ and $2014=2\cdot 19\cdot 53$ are all radioactive, but $2015=5\cdot 13\cdot 31$ is not. How many radioactive numbers have all prime factors less than 30?

Proposed by Evan Chen

17 Let $AXYBZ$ be a convex pentagon inscribed in a circle with diameter $\overline{AB}$. The tangent to the circle at Y intersects lines BX and BZ at L and K, respectively. Suppose that $\overline{AY}$ bisects $\angle LAZ$ and $AY = YZ$. If the minimum possible value of

$$\frac{AK}{AX} + \left(\frac{AL}{AB}\right)^2$$

can be written as $\frac{m}{n} + \sqrt{k}$, where m, n and k are positive integers with $\gcd(m,n) = 1$, compute $m + 10n + 100k$.

Proposed by Evan Chen

18 Find the number of pairs (m, n) of integers with $-2014 \le m, n \le 2014$ such that $x^3 + y^3 = m + 3nxy$ has infinitely many integer solutions (x, y).

Proposed by Victor Wang

19 Find the sum of all positive integers n such that $\tau(n)^2 = 2n$, where $\tau(n)$ is the number of positive integers dividing n.

Proposed by Michael Kural

20 Let ABC be an acute triangle with circumcenter O, and select E on $\overline{AC}$ and F on $\overline{AB}$ so that $\overline{BE} \perp \overline{AC}$, $\overline{CF} \perp \overline{AB}$. Suppose $\angle EOF - \angle A = 90^\circ$ and $\angle AOB - \angle B = 30^\circ$. If the maximum possible measure of $\angle C$ is $\frac{m}{n} \cdot 180^\circ$ for some positive integers m and n with $m < n$ and $\gcd(m, n) = 1$, compute $m + n$.

Proposed by Evan Chen

21 Let $b = \frac{1}{2}(-1 + 3\sqrt{5})$. Determine the number of rational numbers which can be written in the form

$$a_{2014}b^{2014} + a_{2013}b^{2013} + \cdots + a_1 b + a_0$$

where $a_0, a_1, \dots, a_{2014}$ are nonnegative integers less than b.

Proposed by Michael Kural and Evan Chen

22 Let $f(x)$ be a polynomial with integer coefficients such that $f(15)f(21)f(35) - 10$ is divisible by 105. Given $f(-34) = 2014$ and $f(0) \ge 0$, find the smallest possible value of $f(0)$.

Proposed by Michael Kural and Evan Chen

23 Let Γ_1 and Γ_2 be circles in the plane with centers O_1 and O_2 and radii 13 and 10, respectively. Assume $O_1O_2 = 2$. Fix a circle Ω with radius 2, internally tangent to Γ_1 at P and externally tangent to Γ_2 at Q. Let ω be a second variable circle internally tangent to Γ_1 at X and externally tangent to Γ_2 at Y. Line PQ meets Γ_2 again at R, line XY meets Γ_2 again at Z, and lines PZ and XR meet at M.

As ω varies, the locus of point M encloses a region of area $\frac{p}{q}\pi$, where p and q are relatively prime positive integers. Compute $p+q$.

Proposed by Michael Kural

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24 Let $\mathcal{P}$ denote the set of planes in three-dimensional space with positive x, y, and z intercepts summing to one. A point (x,y,z) with $\min\{x,y,z\} > 0$ lies on exactly one plane in $\mathcal{P}$. What is the maximum possible integer value of $\left(\frac{1}{4}x^2+2y^2+16z^2\right)^{-1}$?

Proposed by Sammy Luo

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25 If

$$\sum_{n=1}^{\infty} \frac{\frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{n}}{\binom{n+100}{100}} = \frac{p}{q}$$

for relatively prime positive integers p,q, find $p+q$.

Proposed by Michael Kural

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26 Qing initially writes the ordered pair $(1,0)$ on a blackboard. Each minute, if the pair (a,b) is on the board, she erases it and replaces it with one of the pairs $(2a-b,a)$, $(2a+b+2,a)$ or $(a+2b+2,b)$. Eventually, the board reads $(2014,k)$ for some nonnegative integer k. How many possible values of k are there?

Proposed by Evan Chen

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27 A frog starts at 0 on a number line and plays a game. On each turn the frog chooses at random to jump 1 or 2 integers to the right or left. It stops moving if it lands on a nonpositive number or a number on which it has already landed. If the expected number of times it will jump is $\frac{p}{q}$ for relatively prime positive integers p and q, find $p+q$.

Proposed by Michael Kural

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28 In the game of Nim, players are given several piles of stones. On each turn, a player picks a nonempty pile and removes any positive integer number of stones from that pile. The player who removes the last stone wins, while the first player who cannot move loses.

Alice, Bob, and Chebyshev play a 3-player version of Nim where each player wants to win but avoids losing at all costs (there is always a player who neither wins nor loses). Initially, the piles have sizes 43, 99, x, y, where x and y are positive integers. Assuming that the first player loses when all players play optimally, compute the maximum possible value of xy.

Proposed by Sammy Luo

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29 Let $ABCD$ be a tetrahedron whose six side lengths are all integers, and let N denote the sum of these side lengths. There exists a point P inside $ABCD$ such that the feet from P onto the faces of the tetrahedron are the orthocenter of $\triangle ABC$, centroid of $\triangle BCD$, circumcenter of $\triangle CDA$, and orthocenter of $\triangle DAB$. If $CD = 3$ and $N < 100{,}000$, determine the maximum possible value of N.

Proposed by Sammy Luo and Evan Chen

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30 For a positive integer n, an [i]n-branch[/i] B is an ordered tuple $(S_1, S_2, \dots, S_m)$ of nonempty sets (where m is any positive integer) satisfying $S_1 \subset S_2 \subset \cdots \subset S_m \subseteq \{1, 2, \dots, n\}$. An integer x is said to *appear* in B if it is an element of the last set S_m. Define an [i]n-plant[/i] to be an (unordered) set of n-branches $\{B_1, B_2, \dots, B_k\}$, and call it *perfect* if each of $1, 2, \dots, n$ appears in exactly one of its branches.

Let T_n be the number of distinct perfect n-plants (where $T_0 = 1$), and suppose that for some positive real number x we have the convergence

$$\ln\left(\sum_{n\ge 0} T_n \cdot \frac{(\ln x)^n}{n!}\right) = \frac{6}{29}.$$

If $x = \frac{m}{n}$ for relatively prime positive integers m and n, compute $m + n$.

Proposed by Yang Liu

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1 Carl has a rectangle whose side lengths are positive integers. This rectangle has the property that when he increases the width by 1 unit and decreases the length by 1 unit, the area increases by x square units. What is the smallest possible positive value of x?

Proposed by Ray Li

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2 Suppose (a_n), (b_n), (c_n) are arithmetic progressions. Given that $a_1 + b_1 + c_1 = 0$ and $a_2 + b_2 + c_2 = 1$, compute $a_{2014} + b_{2014} + c_{2014}$.

Proposed by Evan Chen

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3 Let $B = (20, 14)$ and $C = (18, 0)$ be two points in the plane. For every line ℓ passing through B, we color red the foot of the perpendicular from C to ℓ. The set of red points enclose a bounded region of area $\mathcal{A}$. Find $\lfloor \mathcal{A} \rfloor$ (that is, find the greatest integer not exceeding $\mathcal{A}$).

Proposed by Yang Liu

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4 A crazy physicist has discovered a new particle called an emon. He starts with two emons in the plane, situated a distance 1 from each other. He also has a crazy machine which can take any two emons and create a third one in the plane such that the three emons lie at the vertices of an equilateral triangle. After he has five total emons, let P be the product of the $\binom{5}{2} = 10$ distances between the 10 pairs of emons. Find the greatest possible value of P^2.

Proposed by Yang Liu

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5 A crazy physicist has discovered a new particle called an omon. He has a machine, which takes two omons of mass a and b and entangles them; this process destroys the omon with mass a, preserves the one with mass b, and creates a new omon whose mass is $\frac{1}{2}(a+b)$. The physicist can then repeat the process with the two resulting omons, choosing which omon to destroy at every step. The physicist initially has two omons whose masses are distinct positive integers less than 1000. What is the maximum possible number of times he can use his machine without producing an omon whose mass is not an integer?

Proposed by Michael Kural

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6 For an olympiad geometry problem, Tina wants to draw an acute triangle whose angles each measure a multiple of 10°. She doesn't want her triangle to have any special properties, so none of the angles can measure 30° or 60°, and the triangle should definitely not be isosceles.

How many different triangles can Tina draw? (Similar triangles are considered the same.)

Proposed by Evan Chen

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7 Define the function $f(x, y, z)$ by

$$f(x, y, z) = x^{y^z} - x^{z^y} + y^{z^x} - y^{x^z} + z^{x^y}.$$

Evaluate $f(1, 2, 3) + f(1, 3, 2) + f(2, 1, 3) + f(2, 3, 1) + f(3, 1, 2) + f(3, 2, 1)$.

Proposed by Robin Park

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8 Let a and b be randomly selected three-digit integers and suppose $a > b$.
We say that a is *clearly bigger* than b if each digit of a is larger than the corresponding digit of b.
If the probability that a is clearly bigger than b is $\frac{m}{n}$, where m and n are relatively prime integers, compute $m + n$.

Proposed by Evan Chen

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9 Let $N = 2014! + 2015! + 2016! + \cdots + 9999!$. How many zeros are at the end of the decimal representation of N?

Proposed by Evan Chen

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10 Find the sum of the decimal digits of

$$\left\lfloor \frac{51525354555657\ldots979899}{50} \right\rfloor.$$

Here $\lfloor x \rfloor$ is the greatest integer not exceeding x.

Proposed by Evan Chen

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11 Given a triangle ABC, consider the semicircle with diameter $\overline{EF}$ on $\overline{BC}$ tangent to $\overline{AB}$ and $\overline{AC}$. If $BE=1$, $EF=24$, and $FC=3$, find the perimeter of $\triangle ABC$.

Proposed by Ray Li

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12 Let a, b, c be positive real numbers for which

$$\frac{5}{a} = b+c, \quad \frac{10}{b} = c+a, \quad \text{and} \quad \frac{13}{c} = a+b.$$

If $a+b+c = \frac{m}{n}$ for relatively prime positive integers m and n, compute $m+n$.

Proposed by Evan Chen

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13 Two ducks, Wat and Q, are taking a math test with 1022 other ducklings. The test has 30 questions, and the nth question is worth n points. The ducks work independently on the test. Wat gets the nth problem correct with probability $\frac{1}{n^2}$ while Q gets the nth problem correct with probability $\frac{1}{n+1}$. Unfortunately, the remaining ducklings each answer all 30 questions incorrectly.

Just before turning in their test, the ducks and ducklings decide to share answers! On any question which Wat and Q have the same answer, the ducklings change their answers to agree with them. After this process, what is the expected value of the sum of all 1024 scores?

Proposed by Evan Chen

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14 What is the greatest common factor of 12345678987654321 and 12345654321?

Proposed by Evan Chen

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15 Let $\phi = \frac{1+\sqrt{5}}{2}$. A [i]base-$\phi$ number[/i] $(a_na_{n-1}\ldots a_1a_0)_\phi$, where $0 \le a_n, a_{n-1}, \ldots, a_0 \le 1$ are integers, is defined by

$$(a_na_{n-1}\ldots a_1a_0)_\phi = a_n\cdot\phi^n + a_{n-1}\cdot\phi^{n-1} + \ldots + a_1\cdot\phi^1 + a_0.$$

Compute the number of base-ϕ numbers $(b_jb_{j-1}\ldots b_1b_0)_\phi$ which satisfy $b_j \neq 0$ and

$$(b_jb_{j-1}\ldots b_1b_0)_\phi = \underbrace{(100\ldots100)_\phi}_{\text{Twenty } 100's}.$$

Proposed by Yang Liu

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16 Let $OABC$ be a tetrahedron such that $\angle AOB = \angle BOC = \angle COA = 90^\circ$ and its faces have integral surface areas. If $[OAB] = 20$ and $[OBC] = 14$, find the sum of all possible values of $[OCA][ABC]$. (Here $[\triangle]$ denotes the area of $\triangle$.)

Proposed by Robin Park

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17 Let ABC be a triangle with area 5 and $BC = 10$. Let E and F be the midpoints of sides AC and AB respectively, and let BE and CF intersect at G. Suppose that quadrilateral $AEGF$ can be inscribed in a circle. Determine the value of $AB^2 + AC^2$.

Proposed by Ray Li

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18 We select a real number α uniformly and at random from the interval $(0, 500)$. Define

$$S = \frac{1}{\alpha}\sum_{m=1}^{1000}\sum_{n=m}^{1000}\left\lfloor \frac{m+\alpha}{n} \right\rfloor.$$

Let p denote the probability that $S \ge 1200$. Compute $1000p$.

Proposed by Evan Chen

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19 In triangle ABC, $AB = 3$, $AC = 5$, and $BC = 7$. Let E be the reflection of A over $\overline{BC}$, and let line BE meet the circumcircle of ABC again at D. Let I be the incenter of $\triangle ABD$. Given that $\cos^2 \angle AEI = \frac{m}{n}$, where m and n are relatively prime positive integers, determine $m+n$.

Proposed by Ray Li

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20 Let $n = 2188 = 3^7 + 1$ and let $A_0^{(0)}, A_1^{(0)}, ..., A_{n-1}^{(0)}$ be the vertices of a regular n-gon (in that order) with center O . For $i = 1, 2, \dots, 7$ and $j = 0, 1, \dots, n-1$, let $A_j^{(i)}$ denote the centroid of the triangle

$$\triangle A_j^{(i-1)} A_{j+3^{7-i}}^{(i-1)} A_{j+2\cdot 3^{7-i}}^{(i-1)}.$$

Here the subscripts are taken modulo n. If

$$\frac{|OA_{2014}^{(7)}|}{|OA_{2014}^{(0)}|} = \frac{p}{q}$$

for relatively prime positive integers p and q, find $p+q$.

Proposed by Yang Liu

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21 Consider a sequence $x_1, x_2, \cdots x_{12}$ of real numbers such that $x_1 = 1$ and for $n = 1, 2, \dots, 10$ let

$$x_{n+2} = \frac{(x_{n+1}+1)(x_{n+1}-1)}{x_n}.$$

Suppose $x_n > 0$ for $n = 1, 2, \dots, 11$ and $x_{12} = 0$. Then the value of x_2 can be written as $\frac{\sqrt{a}+\sqrt{b}}{c}$ for positive integers a, b, c with $a > b$ and no square dividing a or b. Find $100a + 10b + c$.

Proposed by Michael Kural

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22 Find the smallest positive integer c for which the following statement holds: Let k and n be positive integers. Suppose there exist pairwise distinct subsets S_1, S_2, $\dots$, S_{2k} of $\{1, 2, \dots, n\}$, such that $S_i \cap S_j \neq \varnothing$ and $S_i \cap S_{j+k} \neq \varnothing$ for all $1 \le i, j \le k$. Then $1000k \le c \cdot 2^n$.

Proposed by Yang Liu

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23 For a prime q, let $\Phi_q(x) = x^{q-1} + x^{q-2} + \dots + x + 1$.
Find the sum of all primes p such that $3 \le p \le 100$ and there exists an odd prime q and a positive integer N satisfying

$$\binom{N}{\Phi_q(p)} \equiv \binom{2\Phi_q(p)}{N} \not\equiv 0 \pmod p.$$

Proposed by Sammy Luo

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24 Let $\mathcal{A} = A_0A_1A_2A_3 \cdots A_{2013}A_{2014}$ be a *regular 2014-simplex*, meaning the 2015 vertices of $\mathcal{A}$ lie in 2014-dimensional Euclidean space and there exists a constant $c > 0$ such that $A_iA_j = c$ for any $0 \le i < j \le 2014$. Let $O = (0, 0, 0, \dots, 0)$, $A_0 = (1, 0, 0, \dots, 0)$, and suppose A_iO has length 1 for $i = 0, 1, \dots, 2014$. Set $P = (20, 14, 20, 14, \dots, 20, 14)$. Find the remainder when

$$PA_0^2 + PA_1^2 + \dots + PA_{2014}^2$$

is divided by 10^6.

Proposed by Robin Park

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25 Kevin has a set S of 2014 points scattered on an infinitely large planar gameboard. Because he is bored, he asks Ashley to evaluate

$$x = 4f_4 + 6f_6 + 8f_8 + 10f_{10} + \cdots$$

while he evaluates

$$y = 3f_3 + 5f_5 + 7f_7 + 9f_9 + \cdots,$$

where f_k denotes the number of convex k-gons whose vertices lie in S but none of whose interior points lie in S.

However, since Kevin wishes to one-up everything that Ashley does, he secretly positions the points so that $y - x$ is as large as possible, but in order to avoid suspicion, he makes sure no three points lie on a single line. Find $|y - x|$.

Proposed by Robin Park

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26 Let ABC be a triangle with $AB = 26$, $AC = 28$, $BC = 30$. Let X, Y, Z be the midpoints of arcs BC, CA, AB (not containing the opposite vertices) respectively on the circumcircle of ABC. Let P be the midpoint of arc BC containing point A. Suppose lines BP and XZ meet at M , while lines CP and XY meet at N. Find the square of the distance from X to MN.

Proposed by Michael Kural

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27 Let $p = 2^{16} + 1$ be a prime, and let S be the set of positive integers not divisible by p.
Let $f : S \to \{0, 1, 2, ..., p-1\}$ be a function satisfying

$$f(x)f(y) \equiv f(xy) + f(xy^{p-2}) \pmod p \quad \text{and} \quad f(x+p) = f(x)$$

for all $x, y \in S$.
Let N be the product of all possible nonzero values of $f(81)$.
Find the remainder when when N is divided by p.

Proposed by Yang Liu and Ryan Alweiss

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28 Let S be the set of all pairs (a, b) of real numbers satisfying $1 + a + a^2 + a^3 = b^2(1 + 3a)$ and $1 + 2a + 3a^2 = b^2 - \frac{5}{b}$. Find $A + B + C$, where

$$A = \prod_{(a,b)\in S} a, \quad B = \prod_{(a,b)\in S} b, \quad \text{and} \quad C = \sum_{(a,b)\in S} ab.$$

Proposed by Evan Chen

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29 Let ABC be a triangle with circumcenter O, incenter I, and circumcircle Γ. It is known that $AB = 7$, $BC = 8$, $CA = 9$. Let M denote the midpoint of major arc $\widehat{BAC}$ of Γ, and let D denote the intersection of Γ with the circumcircle of $\triangle IMO$ (other than M). Let E denote the reflection of D over line IO. Find the integer closest to $1000 \cdot \frac{BE}{CE}$.

Proposed by Evan Chen

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30 Let $p = 2^{16} + 1$ be an odd prime. Define $H_n = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n}$. Compute the remainder when

$$(p-1)!\sum_{n=1}^{p-1} H_n \cdot 4^n \cdot \binom{2p-2n}{p-n}$$

is divided by p.

Proposed by Yang Liu
