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Day 1 April 19th

1 Prove that there are infinitely many distinct pairs (a, b) of relatively prime integers $a > 1$ and $b > 1$ such that $a^b + b^a$ is divisible by $a + b$.

2 Let $m_1, m_2, \ldots, m_n$ be a collection of n positive integers, not necessarily distinct. For any sequence of integers $A = (a_1, \ldots, a_n)$ and any permutation $w = w_1, \ldots, w_n$ of $m_1, \ldots, m_n$, define an [i]A-inversion[/i] of w to be a pair of entries w_i, w_j with $i < j$ for which one of the following conditions holds:

-$a_i \geq w_i > w_j$
-$w_j > a_i \geq w_i$, or
-$w_i > w_j > a_i$.

Show that, for any two sequences of integers $A = (a_1, \ldots, a_n)$ and $B = (b_1, \ldots, b_n)$, and for any positive integer k, the number of permutations of $m_1, \ldots, m_n$ having exactly k A-inversions is equal to the number of permutations of $m_1, \ldots, m_n$ having exactly k B-inversions.

3 Let ABC be a scalene triangle with circumcircle Ω and incenter I. Ray AI meets $\overline{BC}$ at D and meets Ω again at M; the circle with diameter $\overline{DM}$ cuts Ω again at K. Lines MK and BC meet at S, and N is the midpoint of $\overline{IS}$. The circumcircles of $\triangle KID$ and $\triangle MAN$ intersect at points L_1 and L_2. Prove that Ω passes through the midpoint of either $\overline{IL_1}$ or $\overline{IL_2}$.

Proposed by Evan Chen

Day 2 April 20th

4 Let P_1, P_2, $\ldots$, P_{2n} be $2n$ distinct points on the unit circle $x^2+y^2 = 1$, other than $(1, 0)$. Each point is colored either red or blue, with exactly n red points and n blue points. Let R_1, R_2, $\ldots$, R_n be any ordering of the red points. Let B_1 be the nearest blue point to R_1 traveling counterclockwise around the circle starting from R_1. Then let B_2 be the nearest of the remaining blue points to R_2 travelling counterclockwise around the circle from R_2, and so on, until we have labeled all of the blue points $B_1, \ldots, B_n$. Show that the number of counterclockwise arcs of the form $R_i \to B_i$ that contain the point $(1, 0)$ is independent of the way we chose the ordering $R_1, \ldots, R_n$ of the red points.

5 Let $\mathbf{Z}$ denote the set of all integers. Find all real numbers $c > 0$ such that there exists a labeling of the lattice points $(x, y) \in \mathbf{Z}^2$ with positive integers for which:

- only finitely many distinct labels occur, and
- for each label i, the distance between any two points labeled i is at least c^i.

Proposed by Ricky Liu

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6 Find the minimum possible value of

$$\frac{a}{b^3+4}+\frac{b}{c^3+4}+\frac{c}{d^3+4}+\frac{d}{a^3+4}$$

given that a, b, c, d are nonnegative real numbers such that $a+b+c+d=4$.

Proposed by Titu Andreescu
