Mathematical Olympiad 2008

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1   Prove that the set $\{1, 2, \cdots, 2007\}$ can be expressed as the union of disjoint subsets A_i for $i = 1, 2, \cdots, 223$ such that each A_i contains nine elements and the sum of all the elements in each A_i is the same.

2   Find the largest integer n for which $\frac{n^{2007} + n^{2006} + \cdots + n^2 + n + 1}{n + 2007}$ is an integer.

3   Let P be a point outside a circle Γ, and let the two tangent lines through P touch Γ at A and B. Let C be on the minor arc AB, and let ray PC intersect Γ again at D. Let ℓ be the line through B and parallel to PA. ℓ intersects AC and AD at E and F, respectively. Prove that B is the midpoint of EF.

4   Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by $f(x) = \frac{2008^{2x}}{2008 + 2008^{2x}}$. Prove that

$$f\left(\frac{1}{2007}\right) + f\left(\frac{2}{2007}\right) + \cdots + f\left(\frac{2005}{2007}\right) + f\left(\frac{2006}{2007}\right) = 1003.$$