Greece Team Selection Test 2014

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1 Let (x_n) $n \geq 1$ be a sequence of real numbers with $x_1 = 1$ satisfying $2x_{n+1} = 3x_n + \sqrt{5x_n^2 - 4}$

a) Prove that the sequence consists only of natural numbers.
b) Check if there are terms of the sequence divisible by 2011.

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2 Find all real non-zero polynomials satisfying $P(x)^3 + 3P(x)^2 = P(x^3) - 3P(-x)$ for all $x \in \mathbb{R}$.

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3 Let ABC be an acute,non-isosceles triangle with $AB < AC < BC$.Let D, E, Z be the midpoints of BC, AC, AB respectively and segments BK, CL are altitudes.In the extension of DZ we take a point M such that the parallel from M to KL crosses the extensions of CA, BA, DE at S, T, N respectively (we extend CA to A-side and BA to A-side and DE to E-side).If the circumcirle (c_1) of $\triangle MBD$ crosses the line DN at R and the circumcirle (c_2) of $\triangle NCD$ crosses the line DM at P prove that $ST \parallel PR$.

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4 Square $ABCD$ is divided into n^2 equal small squares by lines parallel to its sides.A spider starts from A and moving only rightward or upwards,tries to reach C.Every "movement" of the spider consists of k steps rightward and m steps upwards or m steps rightward and k steps upwards(it can follow any possible order for the steps of each "movement").The spider completes l "movements" and afterwards it moves without limitation (it still moves rightwards and upwards only).If $n = m \cdot l$,find the number of the possible paths the spider can follow to reach C.Note that $n, m, k, l \in \mathbb{N}^*$ with $k < m$.
