Romania National Olympiad 2001

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– Grade level 7

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1 Show that there exist no integers a and b such that $a^3+a^2b+ab^2+b^3=2001$.

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2 Let a and b be real, positive and distinct numbers. We consider the set:

$$M=\{ax+by\mid x,y\in\mathbb{R},\ x>0,\ y>0,\ x+y=1\}$$

Prove that:

(i) $\frac{2ab}{a+b}\in M$;

(ii) $\sqrt{ab}\in M$.

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3 We consider a right trapezoid $ABCD$, in which $AB||CD$, $AB>CD$, $AD\perp AB$ and $AD>CD$. The diagonals AC and BD intersect at O. The parallel through O to AB intersects AD in E and BE intersects CD in F. Prove that $CE\perp AF$ if and only if $AB\cdot CD=AD^2-CD^2$.

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4 Consider the acute angle ABC. On the half-line BC we consider the distinct points P and Q whose projections onto the line AB are the points M and N. Knowing that $AP=AQ$ and $AM^2-AN^2=BN^2-BM^2$, find the angle ABC.

– Grade level 8

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1 Determine all real numbers a and b such that $a+b\in\mathbb{Z}$ and $a^2+b^2=2$.

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2 For every rational number $m>0$ we consider the function $f_m:\mathbb{R}\rightarrow\mathbb{R}, f_m(x)=\frac{1}{m}x+m$. Denote by G_m the graph of the function f_m. Let p,q,r be positive rational numbers.

a) Show that if p and q are distinct then $G_p\cap G_q$ is non-empty.

b) Show that if $G_p\cap G_q$ is a point with integer coordinates, then p and q are integer numbers.

c) Show that if p,q,r are consecutive natural numbers, then the area of the triangle determined by intersections of G_p,G_q and G_r is equal to 1.

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3 We consider the points A,B,C,D, not in the same plane, such that $AB\perp CD$ and $AB^2+CD^2=AD^2+BC^2$.

a) Prove that $AC\perp BD$.

b) Prove that if $CD < BC < BD$, then the angle between the planes (ABC) and (ADC) is greater than $60°$.

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4 In the cube $ABCDA'B'C'D'$, with side a, the plane $(AB'D')$ intersects the planes $(A'BC)$, $(A'CD)$, $(A'DB)$ after the lines d_1, d_2 and d_3 respectively.

a) Show that the lines d_1, d_2, d_3 intersect pairwise.

b) Determine the area of the triangle formed by these three lines.

– Grade level 9

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1 Let A be a set of real numbers which verifies:

$$a)\ 1 \in Ab)\ x \in A \implies x^2 \in Ac)\ x^2 - 4x + 4 \in A \implies x \in A$$

Show that $2000 + \sqrt{2001} \in A$.

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2 Let ABC be a triangle $(A = 90°)$ and $D \in (AC)$ such that BD is the bisector of B. Prove that $BC - BD = 2AB$ if and only if

$$\frac{1}{BD} - \frac{1}{BC} = \frac{1}{2AB}$$

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3 Let $n \in \mathbb{N}^*$ and $v_1, v_2, \ldots, v_n$ be vectors in the plane with lengths less than or equal to 1. Prove that there exists $\xi_1, \xi_2, \ldots, \xi_n \in \{-1, 1\}$ such that

$$|\xi_1 v_1 + \xi_2 v_2 + \ldots + \xi_n v_n| \leq \sqrt{2}$$

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4 Determine the ordered systems (x, y, z) of positive rational numbers for which $x + \frac{1}{y}, y + \frac{1}{z}$ and $z + \frac{1}{x}$ are integers.

– Grade level 10

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1 Let a and b be complex non-zero numbers and z_1, z_2 the roots of the polynomials $X^2 + aX + b$. Show that $|z_1 + z_2| = |z_1| + |z_2|$ if and only if there exists a real number $\lambda \geq 4$ such that $a^2 = \lambda b$.

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2 In the tetrahedron $OABC$ we denote by α, β, γ the measures of the angles $\angle BOC$, $\angle COA$, and $\angle AOB$, respectively. Prove the inequality

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma < 1 + 2\cos\alpha\cos\beta\cos\gamma$$

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3 Let m, k be positive integers, $k < m$ and M a set with m elements. Prove that the maximal number of subsets $A_1, A_2, \dots, A_p$ of M for which $A_i \cap A_j$ has at most k elements, for every $1 \le i < j \le p$, equals

$$p_{max} = \binom{m}{0} + \binom{m}{1} + \binom{m}{2} + \dots + \binom{m}{k+1}$$

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4 Let $n \geq 2$ be an even integer and a, b real numbers such that $b^n = 3a + 1$. Show that the polynomial $P(X) = (X^2 + X + 1)^n - X^n - a$ is divisible by $Q(X) = X^3 + X^2 + X + b$ if and only if $b = 1$.

– Grade level 11

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1 Let $f : \mathbb{R} \to \mathbb{R}$ a continuous function, derivable on $R \backslash \{x_0\}$, having finite side derivatives in x_0. Show that there exists a derivable function $g : \mathbb{R} \to \mathbb{R}$, a linear function $h : \mathbb{R} \to \mathbb{R}$ and $\alpha \in \{-1, 0, 1\}$ such that:

$$f(x) = g(x) + \alpha |h(x)|, \ \forall x \in \mathbb{R}$$

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2 We consider a matrix $A \in M_n(\textbf{C})$ with rank r, where $n \geq 2$ and $1 \leq r \leq n-1$.

a) Show that there exist $B \in M_{n,r}(\textbf{C}), C \in M_{r,n}(\textbf{C})$, with $B = C = r$, such that $A = BC$.

b) Show that the matrix A verifies a polynomial equation of degree $r+1$, with complex coefficients.

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3 Let $f : \mathbb{R} \to [0, \infty)$ be a function with the property that $|f(x) - f(y)| \leq |x - y|$ for every $x, y \in \mathbb{R}$.

Show that:

a) If $\lim_{n \to \infty} f(x+n) = \infty$ for every $x \in \mathbb{R}$, then $\lim_{x \to \infty} = \infty$.

b) If $\lim_{n \to \infty} f(x+n) = \alpha, \alpha \in [0, \infty)$ for every $x \in \mathbb{R}$, then $\lim_{x \to \infty} = \alpha$.

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4 The continuous function $f : [0, 1] \to \mathbb{R}$ has the property:

$$\lim_{x \to \infty} n \left(f \left(x + \frac{1}{n} \right) - f(x) \right) = 0$$

for every $x \in [0, 1)$.

Show that:

a) For every $\epsilon > 0$ and $\lambda \in (0, 1)$, we have:

$$\sup \{ x \in [0, \lambda) \mid |f(x) - f(0)| \leq \epsilon x \} = \lambda$$

b) f is a constant function.

– Grade level 12

1 a) Consider the polynomial $P(X) = X^5 \in \mathbb{R}[X]$. Show that for every $\alpha \in \mathbb{R}^*$, the polynomial $P(X+\alpha) - P(X)$ has no real roots.

b) Let $P(X) \in \mathbb{R}[X]$ be a polynomial of degree $n \geq 2$, with real and distinct roots. Show that there exists $\alpha \in \mathbb{Q}^*$ such that the polynomial $P(X+\alpha) - P(X)$ has only real roots.

2 Let A be a finite ring. Show that there exists two natural numbers m, p where $m > p \geq 1$, such that $a^m = a^p$ for all $a \in A$.

3 Let $f : [-1,1] \to \mathbb{R}$ be a continuous function. Show that:

a) if $\int_0^1 f(\sin(x+\alpha))\,dx = 0$, for every $\alpha \in \mathbb{R}$, then $f(x) = 0,\ \forall x \in [-1,1]$.

b) if $\int_0^1 f(\sin(nx))\,dx = 0$, for every $n \in \mathbb{Z}$, then $f(x) = 0,\ \forall x \in [-1,1]$.

4 Let $f : [0,\infty) \to \mathbb{R}$ be a periodical function, with period 1, integrable on $[0,1]$. For a strictly increasing and unbounded sequence $(x_n)_{n\geq 0},\ x_0 = 0$, with $\lim_{n\to\infty}(x_{n+1} - x_n) = 0$, we denote $r(n) = \max\{k \mid x_k \leq n\}$.

a) Show that:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{r(n)} (x_k - x_{k+1}) f(x_k) = \int_0^1 f(x)\,dx$$

b) Show that:

$$\lim_{n\to\infty} \frac{1}{\ln n} \sum_{k=1}^{r(n)} \frac{f(\ln k)}{k} = \int_0^1 f(x)\,dx$$