Romania National Olympiad 2002

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–	Grade level 7
1	Eight card players are seated around a table. One remarks that at some moment, any player and his two neighbours have altogether an odd number of winning cards. Show that any player has at that moment at least one winning card.
2	Prove that any real number $0 < x < 1$ can be written as a difference of two positive and less than 1 irrational numbers.
3	Let $ABCD$ be a trapezium and AB and CD be it's parallel edges. Find, with proof, the set of interior points P of the trapezium which have the property that P belongs to at least two lines each intersecting the segments AB and CD and each dividing the trapezium in two other trapezoids with equal areas.
4	$a)$ An equilateral triangle of sides a is given and a triangle MNP is constructed under the following conditions: $P \in (AB), M \in (BC), N \in (AC)$, such that $MP \perp AB, NM \perp BC$ and $PN \perp AC$. Find the length of the segment MP. $b)$ Show that for any acute triangle ABC one can find points $P \in (AB), M \in (BC), N \in (AC)$ such that $MP \perp AB, NM \perp BC$ and $PN \perp AC$.
–	Grade level 8
1	For any number $n \in \mathbb{N}, n \geq 2$, denote by $P(n)$ the number of pairs (a, b) whose elements are of positive integers such that $$\frac{n}{a} \in (0,1), \quad \frac{a}{b} \in (1,2) \quad \text{and} \quad \frac{b}{n} \in (2,3).$$ $a)$ Calculate $P(3)$. $b)$ Find n such that $P(n) = 2002$.
2	Given real numbers a, c, d show that there exists at most one function $f : \mathbb{R} \to \mathbb{R}$ which satisfies: $$f(ax+c)+d \leq x \leq f(x+d)+c \quad \text{for any } x \in \mathbb{R}$$
3	Let $[ABCDEF]$ be a frustum of a regular pyramid. Let G and G' be the centroids of bases ABC and DEF respectively. It is known that $AB = 36, DE = 12$ and $GG' = 35$. $a)$ Prove that the planes $(ABF), (BCD), (CAE)$ have a common point P, and the planes $(DEC), (EFA), (FDB)$ have a common point P', both situated on GG'. $b)$ Find the length of the segment $[PP']$.

4 The right prism $[A_1A_2A_3\dots A_nA_1'A_2'A_3'\dots A_n'], n\in\mathbb{N}, n\ge 3$, has a convex polygon as its base. It is known that $A_1A_2'\perp A_2A_3', A_2A_3'\perp A_3A_4',\dots A_{n-1}A_n'\perp A_nA_1', A_nA_1'\perp A_1A_2'$. Show that: $a)$ $n=3$; $b)$ the prism is regular.

– Grade level 9

1 Let $ab+bc+ca=1$. Show that

$$\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\ge\sqrt{3}+\frac{ab}{a+b}+\frac{bc}{b+c}+\frac{ca}{c+a}$$

2 Let ABC be a right triangle where $\measuredangle A=90^\circ$ and $M\in(AB)$ such that $\frac{AM}{MB}=3\sqrt{3}-4$. It is known that the symmetric point of $M$with respect to the line GI lies on AC. Find the measure of $\measuredangle B$.

3 Let k and n be positive integers with $n>2$. Show that the equation:

$$x^n-y^n=2^k$$

has no positive integer solutions.

4 Find all functions $f:\mathbb{N}\to\mathbb{N}$ which satisfy the inequality:

$$f(3x+2y)=f(x)f(y)$$

for all non-negative integers x,y.

– Grade level 10

1 Let X,Y,Z,T be four points in the plane. The segments $[XY]$ and $[ZT]$ are said to be *connected*, if there is some point O in the plane such that the triangles OXY and OZT are right-angled at O and isosceles.
Let $ABCDEF$ be a convex hexagon such that the pairs of segments $[AB],[CE]$, and $[BD],[EF]$ are *connected*. Show that the points A,C,D and F are the vertices of a parallelogram and $[BC]$ and $[AE]$ are *connected*.

2 Find all real polynomials f and g, such that:

$$(x^2+x+1)\cdot f(x^2-x+1)=(x^2-x+1)\cdot g(x^2+x+1),$$

for all $x\in\mathbb{R}$.

3 Find all real numbers a, b, c, d, e in the interval $[-2, 2]$, that satisfy:

$$a+b+c+d+e=0$$
$$a^3+b^3+c^3+d^3+e^3=0$$
$$a^5+b^5+c^5+d^5+e^5=10$$

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4 Let $I \subseteq \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$ a function such that:

$$|f(x)-f(y)| \leq |x-y|, \quad \text{for all } x, y \in I.$$

Show that f is monotonic on I if and only if, for any $x, y \in I$, either $f(x) \leq f\left(\frac{x+y}{2}\right) \leq f(y)$ or $f(y) \leq f\left(\frac{x+y}{2}\right) \leq f(x)$.

– Grade level 11

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1 In the Cartesian plane consider the hyperbola

$$\Gamma = \{M(x,y) \in \mathbb{R}^2 | \frac{x^2}{4} - y^2 = 1\}$$

and a conic Γ', disjoint from Γ. Let $n(\Gamma, \Gamma')$ be the maximal number of pairs of points $(A, A') \in \Gamma \times \Gamma'$ such that $AA' \leq BB'$, for any (B, B')
For each $p \in \{0, 1, 2, 4\}$, find the equation of Γ' for which $n(\Gamma, \Gamma') = p$. Justify the answer.

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2 Let $f : \mathbb{R} \to \mathbb{R}$ be a function that has limits at any point and has no local extrema. Show that: $a)$ f is continuous; $b)$ f is strictly monotone.

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3 Let $A \in M_4(C)$ be a non-zero matrix. $a)$ If rank$(A) = r < 4$, prove the existence of two invertible matrices $U, V \in M_4(C)$, such that:

$$UAV = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$$

where I_r is the r-unit matrix. $b)$ Show that if A and A^2 have the same rank k, then the matrix A^n has rank k, for any $n \geq 3$.

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4 Let $f : [0, 1] \to [0, 1]$ be a continuous and bijective function.
Describe the set:

$$A = \{f(x) - f(y) \mid x, y \in [0, 1] \backslash \mathbb{Q}\}$$

You are given the result that [i]there is no one-to-one function between the irrational numbers and $\mathbb{Q}$.[/i]

– Grade level 12

1 Let A be a ring. $a)$ Show that the set $Z(A)=\{a\in A|ax=xa,\text{ for all }x\in A\}$ is a subring of the ring A. $b)$ Prove that, if any commutative subring of A is a field, then A is a field.

2 Let $f:[0,1]\to\mathbb{R}$ be an integrable function such that:

$$0<\left|\int_0^1 f(x)\,\mathrm{d}x\right|\le 1.$$

Show that there exists $x_1\ne x_2, x_1,x_2\in[0,1]$, such that:

$$\int_{x_1}^{x_2} f(x)\,\mathrm{d}x=(x_1-x_2)^{2002}$$

3 Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous and bounded function such that

$$x\int_x^{x+1} f(t)\,\mathrm{d}t=\int_0^x f(t)\,\mathrm{d}t,\quad\text{for any } x\in\mathbb{R}.$$

Prove that f is a constant function.

4 Let K be a field having $q=p^n$ elements, where p is a prime and $n\ge 2$ is an arbitrary integer number. For any $a\in K$, one defines the polynomial $f_a=X^q-X+a$. Show that: $a)$ $f=(X^q-X)^q-(X^q-X)$ is divisible by f_1; $b)$ f_a has at least p^{n-1} essentially different irreducible factors $K[X]$.