Romania National Olympiad 2003

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– Grade level 7

1 Find the maximum number of elements which can be chosen from the set $\{1,2,3,\ldots,2003\}$ such that the sum of any two chosen elements is not divisible by 3.

2 Compute the maximum area of a triangle having a median of length 1 and a median of length 2.

3 For every positive integer n consider

$$A_n=\sqrt{49n^2+0,35n}.$$

(a) Find the first three digits after decimal point of A_1.
(b) Prove that the first three digits after decimal point of A_n and A_1 are the same, for every n.

4 In triangle ABC, P is the midpoint of side BC. Let $M\in(AB)$, $N\in(AC)$ be such that $MN\parallel BC$ and $\{Q\}$ be the common point of MP and BN. The perpendicular from Q on AC intersects AC in R and the parallel from B to AC in T. Prove that:
(a) $TP\parallel MR$;
(b) $\angle MRQ=\angle PRQ$.

Mircea Fianu

– Grade level 8

1 Let m,n be positive integers. Prove that the number 5^n+5^m can be represented as sum of two perfect squares if and only if $n-m$ is even.

Vasile Zidaru

2 In a meeting there are 6 participants. It is known that among them there are seven pairs of friends and in any group of three persons there are at least two friends. Prove that:
(a) there exists a person who has at least three friends;
(b) there exists three persons who are friends with each other.

Valentin Vornicu

3 The real numbers a,b fulfil the conditions
(i) $0<a<a+\frac{1}{2}\le b$;

(ii) $a^{40} + b^{40} = 1$.
Prove that b has the first 12 digits after the decimal point equal to 9.

Mircea Fianu

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4 In tetrahedron $ABCD$, G_1, G_2 and G_3 are barycenters of the faces ACD, ABD and BCD respectively.
(a) Prove that the straight lines BG_1, CG_2 and AG_3 are concurrent.
(b) Knowing that $AG_3 = 8, BG_1 = 12$ and $CG_2 = 20$ compute the maximum possible value of the volume of $ABCD$.

– Grade level 9

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1 Find positive integers a, b if for every $x, y \in [a, b]$, $\frac{1}{x} + \frac{1}{y} \in [a, b]$.

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2 An integer n, $n \geq 2$ is called *friendly* if there exists a family $A_1, A_2, \ldots, A_n$ of subsets of the set $\{1, 2, \ldots, n\}$ such that:
(1) $i \notin A_i$ for every $i = \overline{1, n}$;
(2) $i \in A_j$ if and only if $j \notin A_i$, for every distinct $i, j \in \{1, 2, \ldots, n\}$;
(3) $A_i \cap A_j$ is non-empty, for every $i, j \in \{1, 2, \ldots, n\}$.
Prove that:
(a) 7 is a friendly number;
(b) n is friendly if and only if $n \geq 7$.

Valentin Vornicu

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3 Prove that the midpoints of the altitudes of a triangle are collinear if and only if the triangle is right.

Dorin Popovici

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4 Let P be a plane. Prove that there exists no function $f : P \to P$ such that for every convex quadrilateral $ABCD$, the points $f(A), f(B), f(C), f(D)$ are the vertices of a concave quadrilateral.

Dinu Şerbănescu

– Grade level 10

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1 Let be a tetahedron $OABC$ with $OA \perp OB \perp OC \perp OA$. Show that

$$OH \leq r\left(1 + \sqrt{3}\right),$$

where H is the orthocenter of ABC and r is radius of the inscribed spere of $OABC$.

Valentin Vornicu

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2 Let be five nonzero complex numbers having the same absolute value and such that zero is equal to their sum, which is equal to the sum of their squares. Prove that the affixes of these numbers in the complex plane form a regular pentagon.

Daniel Jinga

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3 Let be a circumcircle of radius 1 of a triangle whose centered representation in the complex plane is given by the affixes of $a, b, c,$ and for which the equation $a + b\cos x + c\sin x = 0$ has a real root in $\left(0, \frac{\pi}{2}\right)$. prove that the area of the triangle is a real number from the interval $\left(1, \frac{1+\sqrt{2}}{2}\right]$.

Gheorghe Iurea

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4 **a)** Prove that the sum of all the elements of a finite union of sets of elements of finite cyclic subgroups of the group of complex numbers, is an integer number.
b) Show that there are finite union of sets of elements of finite cyclic subgroups of the group of complex numbers such that the sum of all its elements is equal to any given integer.

Paltin Ionescu

– Grade level 11

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1 Find the locus of the points M that are situated on the plane where a rhombus $ABCD$ lies, and satisfy:

$$MA \cdot MC + MB \cdot MD = AB^2$$

Ovidiu Pop

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2 Let be eight real numbers $1 \le a_1 < a_2 < a_3 < a_4, x_1 < x_2 < x_3 < x_4$. Prove that

$$\begin{vmatrix} a_1^{x_1} & a_1^{x_2} & a_1^{x_3} & a_1^{x_4} \\ a_2^{x_1} & a_2^{x_2} & a_2^{x_3} & a_2^{x_4} \\ a_3^{x_1} & a_3^{x_2} & a_3^{x_3} & a_3^{x_4} \\ a_4^{x_1} & a_4^{x_2} & a_4^{x_3} & a_4^{x_4} \end{vmatrix} > 0.$$

Marian Andronache, Ion Savu

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3 Let be two functions $f, g : \mathbb{R}_{\ge 0} \longrightarrow \mathbb{R}$ having that properties that f is continuous, g is nondecreasing and unbounded, and for any sequence of rational numbers $(x_n)_{n\ge 1}$ that diverges to ∞, we have

$$1 = \lim_{n\to\infty} f(x_n) g(x_n).$$

Prove that $1 = \lim_{x\to\infty} f(x) g(x)$.

Radu Gologan

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4 Let be a 3×3 real matrix A. Prove the following statements.
a) $f(A) \neq O_3$, for any polynomials $f \in \mathbb{R}[X]$ whose roots are not real.
b) $\exists n \in \mathbb{N} \quad (A + \text{adj}(A))^{2n} = (A)^{2n} + (\text{adj}(A))^{2n} \iff \det(A) = 0$

Laurențiu Panaitopol

– Grade level 12

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1 **a)** Determine the center of the ring of square matrices of a certain dimensions with elements in a given field, and prove that it is isomorphic with the given field.
b) Prove that

$$(\mathcal{M}_n(\mathbb{R}), +, \cdot) \ncong (\mathcal{M}_n(\mathbb{C}), +, \cdot),$$

for any natural number $n \geq 2$.

Marian Andronache, Ion Sava

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2 Let be an odd natural number $n \geq 3$. Find all continuous functions $f : [0,1] \longrightarrow \mathbb{R}$ that satisfy the following equalities.

$$\int_0^1 \left(f\left(\sqrt[k]{x}\right)\right)^{n-k} dx = k/n, \quad \forall k \in \{1, 2, \ldots, n-1\}$$

Titu Andreescu

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3 Let be a continuous function $f : \mathbb{R} \longrightarrow \mathbb{R}$ that has the property that

$$xf(x) \geq \int_0^x f(t)dt,$$

for all real numbers x. Prove that

a) the mapping $x \mapsto \frac{1}{x}\int_0^x f(t)dt$ is nondecreasing on the restrictions $\mathbb{R}_{<0}$ and $\mathbb{R}_{>0}$.

b) if $\int_x^{x+1} f(t)dt = \int_{x-1}^x f(t)dt$, for any real number x, then f is constant.

Mihai Piticari

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4 $i(L)$ denotes the number of multiplicative binary operations over the set of elements of the finite additive group L such that the set of elements of L, along with these additive and multiplicative operations, form a ring. Prove that

a) $i\left(\mathbb{Z}_{12}\right) = 4.$
b) $i(A \times B) \geq i(A)i(B)$, for any two finite commutative groups B and A.
c) there exist two sequences $\left(G_k\right)_{k\geq 1}, \left(H_k\right)_{k\geq 1}$ of finite commutative groups such that

$$\lim_{k\to\infty} \frac{\#G_k}{i\left(G_k\right)} = 0$$

and

$$\lim_{k\to\infty} \frac{\#H_k}{i\left(H_k\right)} = \infty.$$

Barbu Berceanu