Romania National Olympiad 2005

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– Grade level 7

– March 30th

1 Let $ABCD$ be a parallelogram. The interior angle bisector of $\angle ADC$ intersects the line BC in E, and the perpendicular bisector of the side AD intersects the line DE in M. Let $F = AM \cap BC$. Prove that:

a) $DE = AF$;
b) $AD \cdot AB = DE \cdot DM$.

Daniela and Marius Lobaza, Timisoara

2 Let a, b be two integers. Prove that

a) $13 \mid 2a + 3b$ if and only if $13 \mid 2b - 3a$;
b) If $13 \mid a^2 + b^2$ then $13 \mid (2a + 3b)(2b + 3a)$.

Mircea Fianu

3 Let $ABCD$ be a quadrilateral with $AB \parallel CD$ and $AC \perp BD$. Let O be the intersection of AC and BD. On the rays $(OA$ and $(OB$ we consider the points M and N respectively such that $\angle ANC = \angle BMD = 90^\circ$. We denote with E the midpoint of the segment MN. Prove that

a) $\triangle OMN \sim \triangle OBA$;
b) $OE \perp AB$.

Claudiu-Stefan Popa

4 On a circle there are written 2005 non-negative integers with sum 7022. Prove that there exist two pairs formed with two consecutive numbers on the circle such that the sum of the elements in each pair is greater or equal with 8.

After an idea of Marin Chirciu

– Grade level 8

– March 30th

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1 We consider a cube with sides of length 1. Prove that a tetrahedron with vertices in the set of the vertices of the cube has the volume $\frac{1}{6}$ if and only if 3 of the vertices of the tetrahedron are vertices on the same face of the cube.

Dinu Serbanescu

2 For a positive integer n, written in decimal base, we denote by $p(n)$ the product of its digits.

a) Prove that $p(n) \leq n$;
b) Find all positive integers n such that

$$10p(n) = n^2 + 4n - 2005.$$

Eugen Pltnea

3 Let the $ABCA'B'C'$ be a regular prism. The points M and N are the midpoints of the sides BB', respectively BC, and the angle between the lines AB' and BC' is of 60°. Let O and P be the intersection of the lines $A'C$ and AC', with respectively $B'C$ and $C'N$.

a) Prove that $AC' \perp (OPM)$;
b) Find the measure of the angle between the line AP and the plane (OPM).

Mircea Fianu

4 a) Prove that for all positive reals u, v, x, y the following inequality takes place:

$$\frac{u}{x} + \frac{v}{y} \geq \frac{4(uy + vx)}{(x + y)^2}.$$

b) Let $a, b, c, d > 0$. Prove that

$$\frac{a}{b + 2c + d} + \frac{b}{c + 2d + a} + \frac{c}{d + 2a + b} + \frac{d}{a + 2b + c} \geq 1.$$

Traian Tmian

– Grade level 9

– March 30th

1 Let $ABCD$ be a convex quadrilateral with $AD \not\parallel BC$. Define the points $E = AD \cap BC$ and $I = AC \cap BD$. Prove that the triangles EDC and IAB have the same centroid if and only if

$AB \parallel CD$ and $IC^2 = IA \cdot AC$.

Virgil Nicula

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2 Find all functions $f : \mathbb{R} \to \mathbb{R}$ for which

$$x(f(x+1) - f(x)) = f(x),$$

for all $x \in \mathbb{R}$ and

$$|f(x) - f(y)| \leq |x - y|,$$

for all $x, y \in \mathbb{R}$.

Mihai Piticari

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3 Prove that for all positive integers n there exists a single positive integer divisible with 5^n which in decimal base is written using n digits from the set $\{1, 2, 3, 4, 5\}$.

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4 Let $x_1, x_2, \ldots, x_n$ be positive reals. Prove that

$$\frac{1}{1+x_1} + \frac{1}{1+x_1+x_2} + \cdots + \frac{1}{1+x_1+\cdots+x_n} < \sqrt{\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}}.$$

Bogdan Enescu

– Grade level 10

– March 30th

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1 Let n be a positive integer, $n \geq 2$. For each $t \in \mathbb{R}$, $t \neq k\pi$, $k \in \mathbb{Z}$, we consider the numbers

$$x_n(t) = \sum_{k=1}^{n} k(n-k)\cos(tk) \text{ and } y_n(t) = \sum_{k=1}^{n} k(n-k)\sin(tk).$$

Prove that if $x_n(t) = y_n(t) = 0$ if and only if $\tan \frac{nt}{2} = n \tan \frac{t}{2}$.

Constantin Buse

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2 The base $A_1A_2 \ldots A_n$ of the pyramid $VA_1A_2 \ldots A_n$ is a regular polygon. Prove that if

$$\angle VA_1A_2 \equiv \angle VA_2A_3 \equiv \cdots \equiv \angle VA_{n-1}A_n \equiv \angle VA_nA_1,$$

then the pyramid is regular.

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3 a) Prove that there are no one-to-one (injective) functions $f:\mathbb{N}\to\mathbb{N}\cup\{0\}$ such that

$$f(mn)=f(m)+f(n),\ \forall\, m,n\in\mathbb{N}.$$

b) Prove that for all positive integers k there exist one-to-one functions $f:\{1,2,\dots,k\}\to\mathbb{N}\cup\{0\}$ such that $f(mn)=f(m)+f(n)$ for all $m,n\in\{1,2,\dots,k\}$ with $mn\leq k$.

Mihai Baluna

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4 For $\alpha\in(0,1)$ we consider the equation $\{x\{x\}\}=\alpha$.

a) Prove that the equation has rational solutions if and only if there exist $m,p,q\in\mathbb{Z},0<p<q$, $\gcd(p,q)=1$, such that $\alpha=\left(\frac{p}{q}\right)^2+\frac{m}{q}$.
b) Find a solution for $\alpha=\frac{2004}{2005^2}$.

– Grade level 11

– March 30th

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1 Let $n\geq 2$ a fixed integer. We shall call a $n\times n$ matrix A with rational elements a *radical* matrix if there exist an infinity of positive integers k, such that the equation $X^k=A$ has solutions in the set of $n\times n$ matrices with rational elements.

a) Prove that if A is a radical matrix then $\det A\in\{-1,0,1\}$ and there exists an infinity of radical matrices with determinant 1;
b) Prove that there exist an infinity of matrices that are not radical and have determinant 0, and also an infinity of matrices that are not radical and have determinant 1.

After an idea of Harazi

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2 Let $f:[0,1)\to(0,1)$ a continous onto (surjective) function.

a) Prove that, for all $a\in(0,1)$, the function $f_a:(a,1)\to(0,1)$, given by $f_a(x)=f(x)$, for all $x\in(a,1)$ is onto;
b) Give an example of such a function.

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3 Let $X_1,X_2,\dots,X_m$ a numbering of the $m=2^n-1$ non-empty subsets of the set $\{1,2,\dots,n\}$, $n\geq 2$. We consider the matrix $(a_{ij})_{1\leq i,j\leq m}$, where $a_{ij}=0$, if $X_i\cap X_j=\emptyset$, and $a_{ij}=1$ otherwise. Prove that the determinant d of this matrix does not depend on the way the numbering was done and compute d.

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4 Let $f:\mathbb{R}\to\mathbb{R}$ be a convex function.

a) Prove that f is continous;
b) Prove that there exists an unique function $g:[0,\infty)\to\mathbb{R}$ such that for all $x\geq 0$ we have

$$f(x+g(x))=f(g(x))-g(x).$$

– Grade level 12

– March 30th

1 Prove that the group morphisms $f:(\mathbb{C},+)\to(\mathbb{C},+)$ for which there exists a positive λ such that $|f(z)|\leq\lambda|z|$ for all $z\in\mathbb{C}$, have the form

$$f(z)=\alpha z+\beta\overline{z}$$

for some complex α, β.

Cristinel Mortici

2 Let G be a group with m elements and let H be a proper subgroup of G with n elements. For each $x\in G$ we denote $H^x=\{xhx^{-1}\mid h\in H\}$ and we suppose that $H^x\cap H=\{e\}$, for all $x\in G-H$ (where by e we denoted the neutral element of the group G).

a) Prove that $H^x=H^y$ if and only if $x^{-1}y\in H$;
b) Find the number of elements of the set $\bigcup_{x\in G}H^x$ as a function of m and n.

Calin Popescu

3 Let $f:[0,\infty)\to(0,\infty)$ a continous function such that $\lim_{n\to\infty}\int_0^x f(t)dt$ exists and it is finite. Prove that

$$\lim_{x\to\infty}\frac{1}{\sqrt{x}}\int_0^x\sqrt{f(t)}dt=0.$$

Radu Miculescu

4 Let A be a ring with 2^n+1 elements, where n is a positive integer and let

$$M=\{k\in\mathbb{Z}\mid k\geq 2,\ x^k=x,\ \forall\, x\in A\}.$$

Prove that the following statements are equivalent:

a) A is a field;
b) M is not empty and the smallest element in M is 2^n+1.

Marian Andronache