Romania National Olympiad 2007

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by maky, icx, Inequalities Master, mychrom

–	Grade level 11
–	April 11th
1	Let $A, B \in \mathcal{M}_2(\mathbb{R})$ (real 2×2 matrices), that satisfy $A^2+B^2=AB$. Prove that $(AB-BA)^2=O_2$.
2	Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous function, and $a<b$ be two points in the image of f (that is, there exists x, y such that $f(x)=a$ and $f(y)=b$). Show that there is an interval I such that $f(I)=[a,b]$.
3	Let $n\geq 2$ be an integer and denote by H_n the set of column vectors ${}^T(x_1,\ x_2,\ \ldots, x_n)\in\mathbb{R}^n$, such that $\sum\lvert x_i\rvert=1$. Prove that there exist only a finite number of matrices $A\in\mathcal{M}_n(\mathbb{R})$ such that the linear map $f:\mathbb{R}^n\to\mathbb{R}^n$ given by $f(x)=Ax$ has the property $f(H_n)=H_n$. In the contest, the problem was given with a) and b): a) Prove the above for $n=2$; b) Prove the above for $n\geq 3$ as well.
4	Let $f:\mathbb{R}\to\mathbb{R}$ be a differentiable function with continuous derivative, that satisfies $f\big(x+f'(x)\big)=f(x)$. Let's call this property (P). a) Show that if f is a function with property (P), then there exists a real x such that $f'(x)=0$. b) Give an example of a non-constant function f with property (P). c) Show that if f has property (P) and the equation $f'(x)=0$ has at least two solutions, then f is a constant function.
–	Grade level 12
–	April 11th

1 Let $\mathcal{F}$ be the set of functions $f:[0,1]\to\mathbb{R}$ that are differentiable, with continuous derivative, and $f(0)=0$, $f(1)=1$. Find the minimum of $\int_0^1\sqrt{1+x^2}\cdot\left(f'(x)\right)^2\,dx$ (where $f\in\mathcal{F}$) and find all functions $f\in\mathcal{F}$ for which this minimum is attained.

In the contest, this was the b) point of the problem. The a) point was simply "Prove the Cauchy inequality in integral form".

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2 Let $f:[0,1]\to(0,+\infty)$ be a continuous function.

a) Show that for any integer $n\geq 1$, there is a unique division $0=a_0<a_1<\ldots<a_n=1$ such that $\int_{a_k}^{a_{k+1}}f(x)\,dx=\frac{1}{n}\int_0^1 f(x)\,dx$ holds for all $k=0,1,\ldots,n-1$.

b) For each n, consider the a_i above (that depend on n) and define $b_n=\frac{a_1+a_2+\ldots+a_n}{n}$. Show that the sequence (b_n) is convergent and compute it's limit.

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3 Let $n\geq 1$ be an integer. Find all rings $(A,+,\cdot)$ such that all $x\in A\setminus\{0\}$ satisfy $x^{2^n+1}=1$.

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4 Let $n\geq 3$ be an integer and S_n the permutation group. G is a subgroup of S_n, generated by $n-2$ transpositions. For all $k\in\{1,2,\ldots,n\}$, denote by $S(k)$ the set $\{\sigma(k)\ :\ \sigma\in G\}$.

Show that for any k, $|S(k)|\leq n-1$.

– Grade level 9

– April 11th

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1 Let $a,b,c,d\in\mathbb{N}^*$ such that the equation

$$x^2-(a^2+b^2+c^2+d^2+1)x+ab+bc+cd+da=0$$

has an integer solution. Prove that the other solution is integer too and both solutions are perfect squares.

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2 Let ABC be an acute angled triangle and point M chosen differently from A,B,C. Prove that M is the orthocenter of triangle ABC if and only if

$$\frac{BC}{MA}\vec{MA}+\frac{CA}{MB}\vec{MB}+\frac{AB}{MC}\vec{MC}=\vec{0}$$

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3 The plane is divided into strips of width 1 by parallel lines (a strip - the region between two parallel lines). The points from the interior of each strip are coloured with red or white, such that in each strip only one color is used (the points of a strip are coloured with the same color). The points on the lines are not coloured. Show that there is an equilateral triangle of side-length 100, with all vertices of the same colour.

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4 Given a set A and a function $f : A \to A$, denote by $f_1(A) = f(A)$, $f_2(A) = f(f_1(A))$, $f_3(A) = f(f_2(A))$, and so on, ($f_n(A) = f(f_{n-1}(A))$, where the notation $f(B)$ means the set $\{f(x) : x \in B\}$ of images of points from B).
Denote also by $f_\infty(A) = f_1(A) \cap f_2(A) \cap \ldots = \bigcap_{n\geq 1} f_n(A)$.

a) Show that if A is finite, then $f(f_\infty(A)) = f_\infty(A)$.

b) Determine if the above is true for $A = \mathbb{N} \times \mathbb{N}$ and the function

$$f\big((m,n)\big) = \begin{cases} (m+1,n) & \text{if } n \geq m \geq 1 \\ (0,0) & \text{if } m > n \\ (0,n+1) & \text{if } n = 0. \end{cases}$$

– Grade level 10

– April 11th

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1 Show that the equation $z^n + z + 1 = 0$ has a solution with $|z| = 1$ if and only if $n - 2$ is divisble by 3.

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2 Solve the equation

$$2^{x^2+x} + \log_2 x = 2^{x+1}$$

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3 For which integers $n \geq 2$, the number $(n-1)^{n^{n+1}} + (n+1)^{n^{n-1}}$ is divisible by n^n ?

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4 a) For a finite set of natural numbers S, denote by $S + S$ the set of numbers $z = x + y$, where $x, y \in S$. Let $m = |S|$. Show that $|S + S| \leq \frac{m(m+1)}{2}$.

b) Let m be a fixed positive integer. Denote by $C(m)$ the greatest integer $k \geq 1$ for which there exists a set S of m integers, such that $\{1, 2, \ldots, k\} \subseteq S \cup (S + S)$. For example, $C(3) = 8$, with $S = \{1, 3, 4\}$. Show that $\frac{m(m+6)}{4} \leq C(m) \leq \frac{m(m+3)}{2}$.

– Grade level 7

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1 In a triangle ABC, where $a = BC$, $b = CA$ and $c = AB$, it is known that: $a + b - c = 2$ and $2ab - c^2 = 4$. Prove that ABC is an equilateral triangle.

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2 Consider the triangle ABC with $m(\angle BAC = 90^\circ)$ and $AC = 2AB$. Let P and Q be the midpoints of AB and AC,respectively. Let M and N be two points found on the side BC such that $CM = BN = x$. It is also known that $2S[MNPQ] = S[ABC]$. Determine x in function of AB.

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3 Consider the triangle ABC with $m(\angle BAC) = 90^\circ$ and $AB < AC$.Let a point D on the side AC such that: $m(\angle ACB) = m(\angle DBA)$.Let E be a point on the side BC such that $DE \perp BC$.It is known that $BD + DE = AC$. Find the measures of the angles in the triangle ABC.

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4 Let m, n be two natural numbers with $m > 1$ and $2^{2m+1} - n^2 \geq 0$. Prove that:

$$2^{2m+1} - n^2 \geq 7.$$

– Grade level 8

– April 11th

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1 Prove that the number 10^{10} can't be written as the product of two natural numbers which do not contain the digit "0" in their decimal representation.

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2 In a building there are 6018 desks in 2007 rooms, and in every room there is at least one desk. Every room can be cleared dividing the desks in the oher rooms such that in every room is the same number of desks. Find out what methods can be used for dividing the desks initially.

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3 a) In a triangle MNP, the lenghts of the sides are less than 2. Prove that the lenght of the altitude corresponding to the side MN is less than $\sqrt{4 - \frac{MN^2}{4}}$.

b) In a tetrahedron $ABCD$, at least 5 edges have their lenghts less than 2.Prove that the volume of the tetrahedron is less than 1.

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4 Let $ABCD$ be a tetrahedron.Prove that if a point M in a space satisfies the relation:

$$\begin{aligned} MA^2 + MB^2 + CD^2 &= MB^2 + MC^2 + DA^2 \\ &= MC^2 + MD^2 + AB^2 \\ &= MD^2 + MA^2 + BC^2. \end{aligned}$$

then it is found on the common perpendicular of the lines AC and BD.
