Romania National Olympiad 2009

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– Grade level 7

– Grade level 8

– Grade level 9

1 On the sides AB, AC of a triangle ABC, consider the points M, respectively, N such that $M \neq A \neq N$ and $\frac{MB}{MA} \neq \frac{NC}{NA}$. Show that the line MN passes through a point not dependent on M and N.

2 Show that for any four positive real numbers a, b, c, d and four negative real numbers e, f, g, h, the terms $ae + bc, ef + cg, fd + gh, da + hb$ are not all positive.

3 Let be a natural number n, a permutation σ of order n, and n nonnegative real numbers $a_1, a_2, \ldots, a_n$. Prove the following inequality.

$$\left(a_1^2 + a_{\sigma(1)}\right)\left(a_2^2 + a_{\sigma(2)}\right)\cdots\left(a_n^2 + a_{\sigma(n)}\right) \geq \left(a_1^2 + a_1\right)\left(a_2^2 + a_2\right)\cdots\left(a_n^2 + a_n\right)$$

4 Let be two natural numbers $m, n \geq 2$, two increasing finite sequences of real numbers $(a_i)_{1\leq i\leq n}, (b_j)_{1\leq j\leq m}$ and the set

$$\{a_i + b_j | 1 \leq i \leq n, 1 \leq j \leq m\}.$$

Show that the set above has $n + m - 1$ elements if and only if the two sequences above are arithmetic progressions and these have the same ratio.

– Grade level 10

1 **a)** Show that two real numbers $x, y > 1$ chosen so that $x^y = y^x$, are equal or there exists a positive real number $m \neq 1$ such that $x = m^{\frac{1}{m-1}}$ and $y = m^{\frac{m}{m-1}}$.

b) Solve in $(1, \infty)^2$ the equation: $x^y + x^{x^{y-1}} = y^x + y^{y^{x-1}}$.

2 Let be a real number $a \in \left[2 + \sqrt{2}, 4\right]$. Find $\inf_{\substack{z\in\mathbb{C} \\ |z|\leq 1}} \left|z^2 - az + a\right|$.

3 Find all functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ that verify the relation

$$f\left(x^3 + y^3\right) = xf\left(y^2\right) + yf\left(x^2\right),$$

for all real numbers x, y.

4 We say that a natural number $n \geq 4$ is *unusual* if, for any $n \times n$ array of real numbers, the sum of the numbers from any 3×3 compact subarray is negative, and the sum of the numbers from any 4×4 compact subarray is positive.
Find all unusual numbers.

– Grade level 11

1 Let $(t_n)_n$ a convergent sequence of real numbers, $t_n \in (0,1),\ (\forall)n \in \mathbb{N}$ and $\lim_{n\to\infty} t_n \in (0,1)$. Define the sequences $(x_n)_n$ and $(y_n)_n$ by

$$x_{n+1} = t_n x_n + (1-t_n)y_n,\ y_{n+1} = (1-t_n)x_n + t_n y_n,\ (\forall)n \in \mathbb{N}$$

and x_0, y_0 are given real numbers.

a) Prove that the sequences $(x_n)_n$ and $(y_n)_n$ are convergent and have the same limit.

b) Prove that if $\lim_{n\to\infty} t_n \in \{0,1\}$, then the question is false.

2 Let $f : \mathbb{R} \to \mathbb{R}$ a continuous function such that for any $x \in \mathbb{R}$, the limit $\lim_{h\to 0} \left|\frac{f(x+h)-f(x)}{h}\right|$ exists and it is finite. Prove that in any real point, f is differentiable or it has finite one-side derivates, of the same modul, but different signs.

3 Let $A, B \in \mathcal{M}_n(\mathbb{C})$ such that $AB = BA$ and $\det B \neq 0$.

a) If $|\det(A + zB)| = 1$ for any $z \in \mathbb{C}$ such that $|z| = 1$, then $A^n = O_n$.
b) Is the question from a) still true if $AB \neq BA$?

4 Let $f, g, h : \mathbb{R} \to \mathbb{R}$ such that f is differentiable, g and h are monotonic, and $f' = f + g + h$. Prove that the set of the points of discontinuity of g coincides with the respective set of h.

– Grade level 12

1 Find all functions $f \in \mathcal{C}^1[0,1]$ that satisfy $f(1) = -1/6$ and

$$\int_0^1 \left(f'(x)\right)^2 dx \leq 2\int_0^1 f(x)dx.$$

2 **a)** Show that the set of nilpotents of a finite, commutative ring, is closed under each of the operations of the ring.
b) Prove that the number of nilpotents of a finite, commutative ring, divides the number of divisors of zero of the ring.

3 Find the natural numbers $n \geq 2$ which have the property that the ring of integers modulo n has exactly an element that is not a sum of two squares.

4 Find all functions $f : [0,1] \longrightarrow [0,1]$ that are bijective, continuous and have the property that, for any continuous function $g : [0,1] \longrightarrow \mathbb{R}$, the following equality holds.

$$\int_0^1 g\left(f(x)\right) dx = \int_0^1 g(x) dx$$