Romania National Olympiad 2011

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–	Grade level 7	

– Grade level 8

– Grade level 9

1 Let be a natural number n and n real numbers $a_1, a_2, \dots, a_n$ such that

$$a_m + a_{m+1} + \cdots + a_n \geq \frac{(m+n)(n-m+1)}{2}, \quad \forall m \in \{1, 2, \dots, n\}.$$

Prove that $a_1^2 + a_2^2 + \cdots + a_n^2 \geq \frac{n(n+1)(2n+1)}{6}$.

2 Prove that any natural number smaller or equal than the factorial of a natural number n is the sum of at most n distinct divisors of the factorial of n.

3 Let ABC be a triangle, I_a be center of the A-excircle. This excircle intersects the lines AB, AC, at P, respectively, Q. The line PQ intersects the lines I_aB, I_aC at D, respectively, E. Let A_1 be the intersection of DC with BE, and define, analogously, B_1, C_1. Show that AA_1, BB_1, CC_1 are concurrent.

4 Let be a natural number n. Prove that there exists a number $k \in \{0, 1, 2, \dots n\}$ such that the floor of $2^{n+k}\sqrt{2}$ is even.

– Grade level 10

1 Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ a function having the property that

$$|f(x+y) + \sin x + \sin y| \leq 2,$$

for all real numbers x, y.

a) Prove that $|f(x)| \leq 1 + \cos x$, for all real numbers x.
b) Give an example of what f may be, if the interval $(-\pi, \pi)$ is included in its support. (https://en.wikipedia.org/wiki/Support_(mathematics))

2 Find all numbers n for which there exist three (not necessarily distinct) roots of unity of order n whose sum is 1.

3 Let be three positive real numbers a, b, c. Show that the function $f : \mathbb{R} \longrightarrow \mathbb{R}$,

$$f(x) = \frac{a^x}{b^x + c^x} + \frac{b^x}{a^x + c^x} + \frac{c^x}{a^x + b^x},$$

is nondecresing on the interval $[0, \infty)$ and nonincreasing on the interval $(-\infty, 0]$.

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4 **a)** Show that there exists exactly a sequence $(x_n, y_n)_{n\geq 0}$ of pairs of nonnegative integers, that satisfy the property that $(1+\sqrt{33})^n = x_n + y_n\sqrt{33}$, for all nonegative integers n.

b) Having in mind the sequence from **a),** prove that, for any natural prime p, at least one of the numbers y_{p-1}, y_p and y_{p+1} are divisible by p.

– Grade level 11

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1 A row of a matrix belonging to $\mathcal{M}_n(\mathbb{C})$ is said to be *permutable* if no matter how we would permute the entries of that row, the value of the determinant doesn't change. Prove that if a matrix has two *permutable* rows, then its determinant is equal to 0 .

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2 Let $u : [a, b] \to \mathbb{R}$ be a continuous function that has finite left-side derivative $u'_l(x)$ in any point $x \in (a, b]$. Prove that the function u is monotonously increasing if and only if $u'_l(x) \geq 0$, for any $x \in (a, b]$.

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3 Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous and strictly decreasing function with $g(\mathbb{R}) = (-\infty, 0)$. Prove that there are no continuous functions $f : \mathbb{R} \to \mathbb{R}$ with the property that there exists a natural number $k \geq 2$ so that : $\underbrace{f \circ f \circ \ldots \circ f}_{k \text{ times}} = g$.

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4 Let A , $B \in \mathcal{M}_2(\mathbb{C})$ so that : $A^2 + B^2 = 2AB$.
a) Prove that : $AB = BA$.
b) Prove that : tr $(A) =$ tr (B) .

– Grade level 12

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1 Prove that a ring that has a prime characteristic admits nonzero nilpotent elements if and only if its characteristic divides the number of its units.

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2 Let be a continuous function $f : [0, 1] \longrightarrow (0, \infty)$ having the property that, for any natural number $n \geq 2$, there exist $n - 1$ real numbers $0 < t_1 < t_2 < \cdots < t_{n-1} < 1$, such that

$$\int_0^{t_1} f(t)dt = \int_{t_1}^{t_2} f(t)dt = \int_{t_2}^{t_3} f(t)dt = \cdots = \int_{t_{n-2}}^{t_{n-1}} f(t)dt = \int_{t_{n-1}}^{1} f(t)dt.$$

Calculate $\lim_{n\to\infty} \frac{n}{\frac{1}{f(0)} + \sum_{i=1}^{n-1} \frac{1}{f(t_i)} + \frac{1}{f(1)}}$.

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3 The equation $x^{n+1} + x = 0$ admits 0 and 1 as its unique solutions in a ring of order $n \geq 2$. Prove that this ring is a skew field.

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4 Let $f, F : \mathbb{R} \longrightarrow \mathbb{R}$ be two functions such that f is nondecreasing, F admits finite lateral derivates in every point of its domain,

$$\lim_{x \to y^-} f(x) \leq \lim_{x \to y^-} \frac{F(x) - F(y)}{x - y}, \lim_{x \to y^+} f(x) \geq \lim_{x \to y^+} \frac{F(x) - F(y)}{x - y},$$

for all real numbers y, and $F(0) = 0$.

Prove that $F(x) = \int_0^x f(t)dt$, for all real numbers x.
