Romania National Olympiad 2012

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–	IX
1	The altitude $[BH]$ dropped onto the hypotenuse of a triangle ABC intersects the bisectors $[AD]$ and $[CE]$ at Q and P respectively. Prove that the line passing through the midpoints of the segments $[QD]$ and $[PE]$ is parallel to the line AC .
2	Find all functions $f:\mathbb{R}\to\mathbb{R}$ with the following property: for any open bounded interval I, the set $f(I)$ is an open interval having the same length with I .
3	Prove that if $n\ge 2$ is a natural number and $x_1,x_2,\ldots,x_n$ are positive real numbers, then: $$4\left(\frac{x_1^3-x_2^3}{x_1+x_2}+\frac{x_2^3-x_3^3}{x_2+x_3}+\ldots+\frac{x_{n-1}^3-x_n^3}{x_{n-1}+x_n}+\frac{x_n^3-x_1^3}{x_n+x_1}\right)\le\ \le (x_1-x_2)^2+(x_2-x_3)^2+\ldots+(x_{n-1}-x_n)^2+(x_n-x_1$$
4	On a table there are $k\ge 2$ piles having $n_1,n_2,\ldots,n_k$ pencils respectively. A *move* consists in choosing two piles having a and b pencils respectively, $a\ge b$ and transferring b pencils from the first pile to the second one. Find the necessary and sufficient condition for $n_1,n_2,\ldots,n_k$, such that there exists a succession of moves through which all pencils are transferred to the same pile.
–	X
1	Let $M=\{x\in\mathbb{C}\,\|\,\|z\|=1,\ \text{Re}\,z\in\mathbb{Q}\}$. Prove that there exist infinitely many equilateral triangles in the complex plane having all affixes of their vertices in the set M .
2	Let a , b and c be three complex numbers such that $a+b+c=0$ and $\|a\|=\|b\|=\|c\|=1$. Prove that: $$3\le \|z-a\|+\|z-b\|+\|z-c\|\le 4,$$ for any $z\in\mathbb{C}$, $\|z\|\le 1$.
3	Let $a,b\in\mathbb{R}$ with $0<a<b$. Prove that: **a)** $2\sqrt{ab}\le\frac{x+y+z}{3}+\frac{ab}{\sqrt[3]{xyz}}\le a+b$, for $x,y,z\in[a,b]$.

b) $\left\{ \frac{x+y+z}{3} + \frac{ab}{\sqrt[3]{xyz}} \,|\, x, y, z \in [a, b] \right\} = [2\sqrt{ab}, a+b]$.

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4 Let n and m be two natural numbers, $m \geq n \geq 2$. Find the number of injective functions

$$f\colon \{1, 2, \ldots, n\} \to \{1, 2, \ldots, m\}$$

such that there exists a unique number $i \in \{1, 2, \ldots, n-1\}$ for which $f(i) > f(i+1)$.

– XI

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1 Let $f, g\colon [0,1] \to [0,1]$ be two functions such that g is monotonic, surjective and $|f(x) - f(y)| \leq |g(x) - g(y)|$, for any $x, y \in [0,1]$.

a) Prove that f is continuous and that there exists some $x_0 \in [0,1]$ with $f(x_0) = g(x_0)$.
b) Prove that the set $\{x \in [0,1] \,|\, f(x) = g(x)\}$ is a closed interval.

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2 Let n and k be two natural numbers such that $n \geq 2$ and $1 \leq k \leq n-1$. Prove that if the matrix $A \in \mathcal{M}_n(\mathbb{C})$ has exactly k minors of order $n-1$ equal to 0 , then $\det(A) \neq 0$.

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3 Let $A, B \in \mathcal{M}_4(\mathbb{R})$ such that $AB = BA$ and $\det(A^2 + AB + B^2) = 0$. Prove that:

$$\det(A+B) + 3\det(A-B) = 6\det(A) + 6\det(B) .$$

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4 Find all differentiable functions $f\colon [0,\infty) \to [0,\infty)$ for which $f(0) = 0$ and $f'(x^2) = f(x)$ for any $x \in [0,\infty)$.

– XII

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1 Let $f\colon [0,\infty) \to \mathbb{R}$ be a continuous function such that $\int_0^n f(x)f(n-x)\,\mathrm{d}x = \int_0^n f^2(x)\,\mathrm{d}x$, for any natural number $n \geq 1$. Prove that f is a periodic function.

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2 Let $(R, +, \cdot)$ be a ring and let f be a surjective endomorphism of R such that $[x, f(x)] = 0$ for any $x \in R$, where $[a, b] = ab - ba$, $a, b \in R$. Prove that:

a) $[x, f(y)] = [f(x), y]$ and $x[x, y] = f(x)[x, y]$, for any $x, y \in R$;
b) If R is a division ring and f is different from the identity function, then R is commutative.

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3 Let $\mathcal{C}$ be the set of integrable functions $f\colon [0,1] \to \mathbb{R}$ such that $0 \leq f(x) \leq x$ for any $x \in [0,1]$. Define the function $V : \mathcal{C} \to \mathbb{R}$ by

$$V(f) = \int_0^1 f^2(x)\,\mathrm{d}x - \left(\int_0^1 f(x)\,\mathrm{d}x\right)^2 \;,\; f \in \mathcal{C}\,.$$

Determine the following two sets:

a) $\{V(f_a)\,|\,0 \leq a \leq 1\}$, where $f_a(x) = 0$, if $0 \leq x \leq a$ and $f_a(x) = x$, if $a < x \leq 1$;
b) $\{V(f)\,|\,f \in \mathcal{C}\}$.

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4 Let m and n be two nonzero natural numbers. Determine the minimum number of distinct complex roots of the polynomial $\prod_{k=1}^{m}(f+k)$, when f covers the set of n^{th} - degree polynomials with complex coefficients.
