Manhattan Mathematical Olympiad 2002

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by AkshajK

–	Grades 5-6
1	You are given a rectangular sheet of paper and scissors. Can you cut it into a number of pieces all having the same size and shape of a polygon with five sides? What about polygon with seven sides?
2	One out of every seven mathematicians is a philosopher, and one out of every nine philosophers is a mathematician. Are there more philosophers or mathematicians?
3	Let us consider all rectangles with sides of length a, b both of which are whole numbers. Do more of these rectangles have perimeter 2000 or perimeter 2002?
4	Somebody placed digits $1, 2, 3, \ldots, 9$ around the circumference of a circle in an arbitrary order. Reading clockwise three consecutive digits you get a 3-digit whole number. There are nine such 3-digit numbers altogether. Find their sum.
–	Grades 7-8
1	Prove that if an integer n is of the form $4m+3$, where m is another integer, then n is not a sum of two perfect squares (a perfect square is an integer which is the square of some integer).
2	Let us consider the sequence $1, 2, 3, \ldots, 2002$. Somebody choses 1002 numbers from the sequence. Prove that there are two of the chosen numbers which are relatively prime (i.e. do not have any common divisors except 1).
3	The product $1 \cdot 2 \cdot \ldots \cdot n$ is denoted by $n!$ and called *n-factorial*. Prove that the product $$1!2!3!\ldots 49!51!\ldots 100!$$ (the factor $50!$ is missing) is the square of an integer number.
4	Find six points $A_1, A_2, \ldots, A_6$ in the plane, such that for each point $A_i, i = 1, 2, \ldots, 6$ there are exactly three of the remaining five points exactly 1 cm from A_i.
–	Grades 9-12

1 Famous French mathematician Pierre Fermat believed that all numbers of the form $F_n = 2^{2^n} + 1$ are prime for all non-negative integers n. Indeed, one can check that $F_0 = 3$, $F_1 = 5$, $F_2 = 17$, $F_3 = 257$ are all prime.

a) Prove that F_5 is divisible by 641. (Hence Fermat was wrong.)
b) Prove that if $k \neq n$ then F_k and F_n are relatively prime (i.e. they do not have any common divisor except 1)
(Notice: using b) one can prove that there are infinitely many prime numbers)

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2 Prove that for any sequence $a_1, a_2, \dots, a_{2002}$ of non-negative integers written in the usual decimal notation with $a_1 > 0$ there exists an integer n such that n^2 starts with digits $a_1, a_2, \dots, a_{2002}$ (in this order).

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3 Prove that for any polygon with all equal angles and for any interior point A, the sum of distances from A to the sides of the polygon does not depend on the position of A.

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4 A triangle has sides with lengths a, b, c such that

$$a^2 + b^2 = 5c^2$$

Prove that medians to the sides of lengths a and b are perpendicular.
