Nordic 2012

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1 The real numbers a, b, c are such that $a^2 + b^2 = 2c^2$, and also such that $a \neq b, c \neq -a, c \neq -b$. Show that

$$\frac{(a+b+2c)(2a^2-b^2-c^2)}{(a-b)(a+c)(b+c)}$$

is an integer.

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2 Given a triangle ABC, let P lie on the circumcircle of the triangle and be the midpoint of the arc BC which does not contain A. Draw a straight line l through P so that l is parallel to AB. Denote by k the circle which passes through B, and is tangent to l at the point P. Let Q be the second point of intersection of k and the line AB (if there is no second point of intersection, choose $Q = B$). Prove that $AQ = AC$.

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3 Find the smallest positive integer n, such that there exist n integers $x_1, x_2, \ldots, x_n$ (not necessarily different), with $1 \leq x_k \leq n$, $1 \leq k \leq n$, and such that

$$x_1 + x_2 + \cdots + x_n = \frac{n(n+1)}{2}, \quad \text{and } x_1 x_2 \cdots x_n = n!,$$

but $\{x_1, x_2, \ldots, x_n\} \neq \{1, 2, \ldots, n\}$.

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4 The number 1 is written on the blackboard. After that a sequence of numbers is created as follows: at each step each number a on the blackboard is replaced by the numbers $a-1$ and $a+1$; if the number 0 occurs, it is erased immediately; if a number occurs more than once, all its occurrences are left on the blackboard. Thus the blackboard will show 1 after 0 steps; 2 after 1 step; $1, 3$ after 2 steps; $2, 2, 4$ after 3 steps, and so on. How many numbers will there be on the blackboard after n steps?
