W. Rudin, Principles of Mathematical Analysis, 3rd Edition
www.artofproblemsolving.com/community/c444062
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1. Let f be defined for all real x, and suppose that

$$|f(x) - f(y)| \leq (x-y)^2$$

for all x and y. Prove that f is constant.
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Starting at $f(0)$, we show that $f(x) = f(0)$ for all $x \in R^1$. Consider the case $x > 0$, and for $n = 1, 2 \ldots$, let $\{x_0, x_1, x_2, \ldots, x_n\}$ be a *partition* of the interval $[0, x]$ with $x_0 = 0$, $x_n = x$, and $x_{j+1} - x_j = \frac{x}{n}$ for $j \in \{0, 1, 2, \ldots, n-1\}$. Then

$$\begin{aligned}|f(x) - f(0)| &= \left|\sum_{j=0}^{n-1} f(x_{j+1}) - f(x_j)\right| \\ &\leq \sum_{j=0}^{n-1} |f(x_{j+1}) - f(x_j)| \\ &\leq \sum_{j=0}^{n-1} (x_{j+1} - x_j)^2 \\ &= \sum_{j=0}^{n-1} \left(\frac{x}{n}\right)^2 \\ &= \frac{x^2}{n}\end{aligned}$$

where the third line of the above inequality follows by assumption. Since $\frac{x^2}{n} \to 0$ as $n \to \infty$, we conclude that $|f(x) - f(0)| = 0$ or $f(x) = f(0)$. The other case $x < 0$ is in a similar way. Hence f is constant.

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2. Suppose $f'(x) > 0$ in (a, b). Prove that f is strictly increasing in (a, b), and let g be its inverse function. Prove that g is differentiable, and that

$$g'(f(x)) = \frac{1}{f'(x)} \quad (a < x < b)$$

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(i) By the mean value theorem (Theorem 5.10), for $x, y \in (a, b)$ with $x < y$,

$$f(y) - f(x) = f'(c)(y - x) > 0$$

for some $c \in (x, y)$. So f is strictly increasing in (a, b).
(ii) Let $g : f(a, b) \to (a, b)$ be the inverse function of f, i.e., $g(f(x)) = x$ for all $x \in (a, b)$. We now show that

$$g'(y) = \lim_{z \to y} \frac{g(z) - g(y)}{z - y}$$

exists for all $y \in f(a, b)$. Put $y = f(x)$ and $z = f(t)$, where $x, t \in (a, b)$, then since f is continuous (by Theorem 5.2), so is g (by Theorem 4.17), and $z \to y$ implies $t \to x$. It follows that

$$\begin{aligned}\lim_{z \to y} \frac{g(z) - g(y)}{z - y} &= \lim_{t \to x} \frac{g(f(t)) - g(f(x))}{f(t) - f(x)} \\ &= \lim_{t \to x} \frac{t - x}{f(t) - f(x)} \\ &= \lim_{t \to x} \frac{1}{\dfrac{f(t) - f(x)}{t - x}} \\ &= \frac{1}{\lim\limits_{t \to x} \dfrac{f(t) - f(x)}{t - x}} \\ &= \frac{1}{f'(x)}\end{aligned}$$

Since $f' > 0$, we see that $g'(y)$ is defined for all $y \in f(a, b)$. Hence g is differentiable. It is also clear that $g'(f(x)) = \frac{1}{f'(x)}$ for all $x \in (a, b)$.

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3. Suppose g is a real function on R^1, with bounded derivative (say $|g'| \leq M$). Fix $\varepsilon > 0$, and define $f(x) = x + \varepsilon g(x)$. Prove that f is one-to-one if ε is small enough. (A set of admissible value of ε can be determined which depends only on M.)
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If $0 < \varepsilon < \frac{1}{M}$, then $g' \geq -M$ implies

$$f'(x) = 1 + \varepsilon g'(x) \geq 1 - \varepsilon M > 0$$

Thus by Exercise 2, f is strictly increasing, and hence f is one-to-one.

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4. If

$$C_0 + \frac{C_1}{2} + \cdots + \frac{C_{n-1}}{n} + \frac{C_n}{n+1} = 0$$

where $C_0, \ldots, C_n$ are real constants, prove that the equation

$$C_0 + C_1 x + \cdots + C_{n-1} x^{n-1} + C_n x^n = 0$$

has at least one real root between 0 and 1.
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Let $f(x) = C_0 x + \frac{C_1}{2}x^2 + \cdots + \frac{C_n}{n+1}x^{n+1}$, then clearly f is a differentiable real function (since $C_0, C_1, \ldots, C_n$ are real constants). Moreover, we have $f(0) = 0$ and

$$f(1) = C_0 + \frac{C_1}{2} + \cdots + \frac{C_{n-1}}{n} + \frac{C_n}{n+1} = 0$$

By the mean value theorem, there exists a point $x_0 \in (0,1)$ such that $f'(x_0) = \frac{f(1)-f(0)}{1-0} = 0$, i.e.,

$$C_0 + C_1 x_0 + \cdots + C_{n-1}x_0^{n-1} + C_n x_0^n = 0$$

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5. Suppose f is defined and differentiable for every $x > 0$, and $f'(x) \to 0$ as $x \to +\infty$. Put $g(x) = f(x+1) - f(x)$. Prove that $g(x) \to 0$ as $x \to +\infty$.
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By the mean value theorem, for $x > 0$, let $c(x)$ be a real number such that $c(x) \in (x, x+1)$ and $f'(c(x)) = f(x+1) - f(x)$. Then $c(x) \to +\infty$ as $x \to +\infty$, which implies

$$g(x) = f'(c(x)) \to 0$$

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6. Suppose
(a) f is continuous for $x \geq 0$,
(b) $f'(x)$ exists for $x > 0$,
(c) $f(0) = 0$,
(d) f' is monotonically increasing.
Put

$$g(x) = \frac{f(x)}{x} \quad (x > 0)$$

and prove that g is monotonically increasing.
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Since $g'(x) = \frac{xf'(x) - f(x)}{x^2}$ where $x > 0$, by Theorem 5.11(a), it suffices to show that

$$xf'(x) - f(x) \geq 0$$

for all $x > 0$. Since conditions (a), (b), and (c) hold, by the mean value theorem,

$$f(x) = f(x) - f(0) = f'(c)(x - 0) = xf'(c)$$

for some $c \in (0, x)$. Since $c < x$ here, by condition (d), $xf'(c) \leq xf'(x)$. So we conclude that $f(x) \leq xf'(x)$.

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7. Suppose $f'(x)$, $g'(x)$ exist, $g'(x) \neq 0$, and $f(x) = g(x) = 0$. Prove that

$$\lim_{t \to x} \frac{f(t)}{g(t)} = \frac{f'(x)}{g'(x)}$$

(This holds also for complex functions.)
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We have

$$\lim_{t \to x} \frac{f(t)}{g(t)} = \lim_{t \to x} \frac{\frac{f(t)-f(x)}{t-x}}{\frac{g(t)-g(x)}{t-x}} = \frac{\lim_{t \to x} \frac{f(t)-f(x)}{t-x}}{\lim_{t \to x} \frac{g(t)-g(x)}{t-x}} = \frac{f'(x)}{g'(x)}$$

where the first equality shown above follows by $f(x) = g(x) = 0$.

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8. Suppose f' is continuous on $[a, b]$ and $\varepsilon > 0$. Prove that there exists $\delta > 0$ such that

$$\left| \frac{f(t)-f(x)}{t-x} - f'(x) \right| < \varepsilon$$

whenever $0 < |t - x| < \delta$, $a \leq x \leq b$, $a \leq t \leq b$. (This could be expressed by saying that f is *uniformly differentiable* on $[a, b]$ if f' is continuous on $[a, b]$.) Does this hold for vector-valued functions too?
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(i) By the mean-valued theorem, there exists $c(t)$ between x and t such that $\frac{f(t)-f(x)}{t-x} = f'(c(t))$. Since f' is continuous on $[a, b]$, and since $c(t) \to x$ as $t \to x$ (by the *squeeze theorem* (`https://en.wikipedia.org/wiki/Squeeze_theorem`)), we have

$$\frac{f(t)-f(x)}{t-x} = f'(c(t)) \to f'(x)$$

as $t \to x$. This completes the proof.
(ii) Yes. Provided that each component of a given vector-valued function has continuous derivative on an interval.

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9. Let f be a continuous real function on R^1, of which it is known that $f'(x)$ exists for all $x \neq 0$ and that $f'(x) \to 3$ as $x \to 0$. Does it follow that $f'(0)$ exists?
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Yes. Note that $f'(x) \to 3$ as $x \to 0$, so by L'Hospital's rule,

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x} = \lim_{x \to 0} f'(x) = 3$$

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10. Suppose f and g are complex differentiable functions on $(0,1)$, $f(x) \to 0$, $g(x) \to 0$, $f'(x) \to A$, $g'(x) \to B$ as $x \to 0$, where A and B are complex numbers, $B \neq 0$. Prove that

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \frac{A}{B}$$

Compare with Example 5.18. *Hint:*

$$\frac{f(x)}{g(x)} = \left\{ \frac{f(x)}{x} - A \right\} \cdot \frac{x}{g(x)} + A \cdot \frac{x}{g(x)}$$

Apply Theorem 5.13 to the real and imaginary parts of $f(x)/x$ and $g(x)/x$.
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Following the hint, write $f = f_1 + if_2$, where f_1 and f_2 are real functions on $(0,1)$. Then $f_1(x) \to 0$ and $f_1'(x) \to \mathrm{Re}(A)$ as $x \to 0$, and by L'Hospital's rule,

$$\lim_{x\to 0} \mathrm{Re}\left[\frac{f(x)}{x}\right] = \lim_{x\to 0} \frac{f_1(x)}{x} = \lim_{x\to 0} f_1'(x) = \mathrm{Re}(A)$$

Similarly we have $\mathrm{Im}\left[\frac{f(x)}{x}\right] \to \mathrm{Im}(A)$, $\mathrm{Re}\left[\frac{g(x)}{x}\right] \to \mathrm{Re}(B)$, and $\mathrm{Im}\left[\frac{g(x)}{x}\right] \to \mathrm{Im}(B)$ as $x \to 0$. So

$$\begin{aligned}\lim_{x\to 0} \frac{f(x)}{x} &= \lim_{x\to 0}\left(\mathrm{Re}\left[\frac{f(x)}{x}\right] + i\,\mathrm{Im}\left[\frac{f(x)}{x}\right]\right)\\ &= \lim_{x\to 0} \mathrm{Re}\left[\frac{f(x)}{x}\right] + i \lim_{x\to 0} \mathrm{Im}\left[\frac{f(x)}{x}\right]\\ &= \mathrm{Re}(A) + i\,\mathrm{Im}(A)\\ &= A\end{aligned}$$

and similarly $\lim_{x\to 0} \frac{g(x)}{x} = B$ or the limit of its reciprocal

$$\lim_{x\to 0} \frac{x}{g(x)} = \lim_{x\to 0} \frac{1}{\frac{g(x)}{x}} = \frac{1}{\lim_{x\to 0} \frac{g(x)}{x}} = \frac{1}{B}$$

exists, since $B \neq 0$. The result will therefore follow.

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11. Suppose f is defined in a neighborhood of x, and suppose $f''(x)$ exists. Show that

$$\lim_{h\to 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2} = f''(x)$$

Show by an example that the limit may exist even $f''(x)$ does not.
Hint: Use Theorem 5.13.
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(i) By L'Hospital's rule,

$$\begin{aligned}
&\lim_{h\to 0}\frac{f(x+h)+f(x-h)-2f(x)}{h^2}\\
&=\lim_{h\to 0}\frac{f'(x+h)-f'(x-h)}{2h}\\
&=\frac{1}{2}\lim_{h\to 0}\left[\frac{f'(x+h)-f'(x)}{h}+\frac{f'(x)-f'(x-h)}{h}\right]\\
&=\frac{1}{2}\left[\lim_{h\to 0}\frac{f'(x+h)-f'(x)}{h}+\lim_{h\to 0}\frac{f'(x)-f'(x-h)}{h}\right]\\
&=\frac{1}{2}\left[f''(x)+f''(x)\right]\\
&=f''(x)
\end{aligned}$$

(ii) For example, let

$$f(x)=\begin{cases} x+1 & x<0\\ 0 & x=0\\ x-1 & x>0\end{cases}$$

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12. If $f(x)=|x|^3$, compute $f'(x)$, $f''(x)$ for all real x, and show that $f^{(3)}(0)$ does not exist.

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(i) For $x>0$, $f'(x)=\left(x^3\right)'=3x^2$. For $x<0$, $f'(x)=\left(-x^3\right)'=-3x^2$. Finally,

$$f'(0)=\lim_{h\to 0}\frac{|h|^3}{h}=\lim_{h\to 0}\operatorname{sgn}(h)h^2=0$$

where

$$\operatorname{sgn}(h)=\begin{cases} 1 & h>0\\ -1 & h<0\end{cases}$$

(ii) For $x>0$, $f''(x)=\left(3x^2\right)'=6x$. For $x<0$, $f''(x)=\left(-3x^2\right)'=-6x$. Finally, use part (i),

$$f''(0)=\lim_{h\to 0}\frac{f'(h)}{h}=\lim_{h\to 0}\frac{\operatorname{sgn}(h)3h^2}{h}=\operatorname{sgn}(h)3\lim_{h\to 0}h=0$$

(iii) To show that $f^{(3)}(0)$ does not exist, just write

$$\frac{f''(h)-f''(0)}{h}=\frac{f''(h)}{h}=\frac{\operatorname{sgn}(h)6h}{h}=6\operatorname{sgn}(h)$$

The result then follows.

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13. Suppose a and x are real numbers, $c > 0$, and f is defined on $[-1, 1]$ by

$$f(x) = \begin{cases} x^a \sin\left(|x|^{-c}\right) & \text{(if } x \neq 0) \\ 0 & \text{(if } x = 0) \end{cases}$$

Prove the following statements:
(a) f is continuous if and only if $a > 0$.
(b) $f'(0)$ exists if and only if $a > 1$.
(c) f' is bounded if and only if $a \geq 1 + c$.
(d) f' is continuous if and only if $a > 1 + c$.
(e) $f''(0)$ exists if and only if $a > 2 + c$.
(f) f'' is bounded if and only if $a \geq 2 + 2c$.
(g) f'' is continuous if and only if $a > 2 + 2c$.
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(a) It is clear that f is continuous at any nonzero point x, so we only consider the point $x = 0$. If $a > 0$, then

$$|f(x)| = \left|x^a \sin\left(|x|^{-c}\right)\right| \leq |x^a| \to 0 \quad \text{as } x \to 0$$

By squeezing, $f(x) \to 0$ as $x \to 0$, and hence f is continuous at 0. Conversely, if $a \leq 0$, let $x_n = \left(2n\pi + \frac{\pi}{2}\right)^{-\frac{1}{c}}$ for $n = 1, 2, \ldots$. Then $x_n \to 0$ as $n \to \infty$, but $-\frac{a}{c} \geq 0$ implies

$$f(x_n) = \left(2n\pi + \frac{\pi}{2}\right)^{-\frac{a}{c}} \geq 1 \quad \text{for all } n$$

This shows that $\lim_{n\to\infty} f(x_n) \neq 0 = f(0)$, and hence f is not continuous at 0.
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(b) If $a > 1$ or $a - 1 > 0$, then

$$\left|\frac{f(x) - f(0)}{x}\right| = \left|\frac{x^a \sin\left(|x|^{-c}\right)}{x}\right| \leq \left|x^{a-1}\right| \to 0 \quad \text{as } x \to 0$$

By squeezing, $f'(0) = 0$. Conversely, if $a \leq 1$ or $a - 1 \leq 0$, let $x_n = \left(2n\pi + \frac{\pi}{2}\right)^{-\frac{1}{c}}$ and $y_n = (2n\pi)^{-\frac{1}{c}}$ for $n = 1, 2, \ldots$. Then $x_n \to 0$ and $y_n \to 0$ as $n \to \infty$, but $-\frac{a-1}{c} \geq 0$ implies

$$\frac{f(x_n) - f(0)}{x_n} = \left(2n\pi + \frac{\pi}{2}\right)^{-\frac{a-1}{c}} \geq 1 \quad \text{for all } n$$

Also, we have $\frac{f(y_n)-f(0)}{y_n} = 0$ for all n. This gives

$$\lim_{n\to\infty} \frac{f(x_n) - f(0)}{x_n} \neq 0 = \lim_{n\to\infty} \frac{f(y_n) - f(0)}{y_n}$$

and hence $f'(0)$ does not exist.
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(c)
—–
(d)
—–
(e)
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(f)
—–
(g)

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14. Let f be a differentiable real function defined on (a, b). Prove that f is convex if and only if f' is monotonically increasing. Assume next that $f''(x)$ exists for every $x \in (a, b)$, and prove that f is convex if and only if $f''(x) \geq 0$ for all $x \in (a, b)$.
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(i) If f is convex. For $x, y, t \in (a, b)$ with $x < t < y$, and let $c = \frac{f(y)-f(x)}{y-x}$. By Exercise 4.23,

$$c \geq \frac{f(t) - f(x)}{t - x} \quad \text{and} \quad c \leq \frac{f(y) - f(t)}{y - t}$$

Since f is differentiable, we have $c \geq f'(x)$ and $c \leq f'(y)$ as $t \to x$ and $t \to y$, respectively. Hence $f'(x) \leq f'(y)$. Conversely, if f' is monotonically increasing, given $x, y \in (a, b)$ with $x < y$, and $\lambda \in (0, 1)$. Put $t = \lambda x + (1 - \lambda)y$, then

$$t - x = (1 - \lambda)(y - x) \quad \text{and} \quad y - t = \lambda(y - x)$$

Since f' is monotonically increasing, applying the mean value theorem,

$$\frac{f(t) - f(x)}{t - x} = f'(c_1) \leq f'(c_2) = \frac{f(y) - f(t)}{y - t}$$

for some $c_1 \in (x, t)$ and $c_2 \in (t, y)$. It follows that

$$\lambda \left[f(t) - f(x)\right] \leq (1 - \lambda) \left[f(y) - f(t)\right]$$

or

$$f(t) \leq \lambda f(x) + (1 - \lambda)f(y)$$

(ii) We first show that f' is monotonically increasing if and only if $f''(x) \geq 0$ for all $x \in (a, b)$. The sufficiency part is the result of Theorem 5.11(a). To show the necessity part, given $x \in (a, b)$, observe that the fraction $\frac{f'(t)-f'(x)}{t-x}$ is always nonnegative, for all $t \in (a, b)$ with $t \neq x$, since f' is monotonically increasing. Thus by definition,

$$f''(x) = \lim_{t \to x} \frac{f'(t) - f'(x)}{t - x} \geq 0$$

Hence, f is convex if and only if f' is monotonically increasing [by part (i)], if and only if $f''(x) \geq 0$ for all $x \in (a, b)$.

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15. Suppose $a \in R^1$, f is a twice-differentiable real function on (a, ∞), and M_0, M_1, M_2 are the least upper bounds of $|f(x)|$, $|f'(x)|$, $|f''(x)|$, respectively, on (a, ∞). Prove that

$$M_1^2 \leq 4M_0M_2$$

Hint: If $h > 0$, Taylor's theorem shows that

$$f'(x) = \frac{1}{2h}\left[f(x+2h) - f(x)\right] - hf''(\xi)$$

for some $\xi \in (x, x+2h)$. Hence

$$\left|f'(x)\right| \leq hM_2 + \frac{M_0}{h}$$

To show that $M_1^2 = 4M_0M_2$ can actually happen, take $a = -1$, define

$$f(x) = \begin{cases} 2x^2 - 1 & (-1 < x < 0) \\ \dfrac{x^2-1}{x^2+1} & (0 \leq x < \infty) \end{cases}$$

and show that $M_0 = 1$, $M_1 = 4$, $M_2 = 4$.
Does $M_1^2 \leq 4M_0M_2$ hold for vector-valued functions too?
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(i) First, note that M_0M_2 be of the form $0 \cdot \infty$ is exceptional. Now we consider the following four cases.
Case 1: $M_0 = \infty$ or $M_2 = \infty$. The result is trivial.
Case 2: $M_0 = 0$. Then $f(x) = 0$, and then $f'(x) = f''(x) = 0$ for all $x \in (a, \infty)$. It follows that $M_1 = M_2 = 0$, and the result is also trivial.
Case 3: $0 < M_0 < \infty$ and $M_2 = 0$. Then $f''(x) = 0$, and then $f'(x) = c$ and $f(x) = cx + d$ for some constants $c, d \in R$, for all $x \in (a, \infty)$. Since M_0 is finite, we need $c = 0$ and then $M_1 = 0$. The result therefore follows.
Case 4: $0 < M_0 < \infty$ and $0 < M_2 < \infty$. Following the hint, use Taylor's theorem (Theorem 5.15), for $x \in (a, \infty)$ and $h > 0$ there exists $\xi \in (x, x+2h)$ such that

$$\begin{aligned} f(x+2h) &= f(x) + f'(x) \cdot 2h + \frac{f''(\xi)}{2} \cdot (2h)^2 \\ &= f(x) + f'(x) \cdot 2h + f''(\xi) \cdot 2h^2 \end{aligned}$$

Thus

$$f'(x) = \frac{1}{2h}\left[f(x+2h) - f(x)\right] - hf''(\xi)$$

and hence

$$|f'(x)| \le hM_2 + \frac{M_0}{h}$$

By assumption, we may put $h = \sqrt{\frac{M_0}{M_2}}$ and the inequality becomes

$$|f'(x)| \le 2\sqrt{M_0 M_2}$$

implying $M_1 \le 2\sqrt{M_0 M_2}$ or $M_1^2 \le 4M_0 M_2$.
(ii) By some calculation,

$$(-1 < x < 0) \begin{cases} f'(x) = 4x \\ f''(x) = 4 \end{cases}$$

$$(0 < x < \infty) \begin{cases} f'(x) = \dfrac{4x}{(x^2+1)^2} \\ f''(x) = \dfrac{-4(3x^2-1)}{(x^2+1)^3} \end{cases}$$

To find $f'(0)$ and $f''(0)$, calculate

$$\lim_{t \to 0+} \frac{f(t) - f(0)}{t} = \lim_{t \to 0+} \frac{\frac{t^2-1}{t^2+1} + 1}{t} = \lim_{t \to 0+} \frac{2t}{t^2+1} = 0$$

$$\lim_{t \to 0-} \frac{f(t) - f(0)}{t} = \lim_{t \to 0-} \frac{(2t^2-1)+1}{t} = \lim_{t \to 0-} 2t = 0$$

Then $f'(0) = 0$. Next,

$$\lim_{t \to 0+} \frac{f'(t) - f'(0)}{t} = \lim_{t \to 0+} \frac{\frac{4t}{(t^2+1)^2}}{t} = \lim_{t \to 0+} \frac{4}{(t^2+1)^2} = 4$$

$$\lim_{t \to 0-} \frac{f'(t) - f'(0)}{t} = \lim_{t \to 0-} \frac{4t}{t} = 4$$

Then $f''(0) = 4$. Now, $-1 \le f(x) < 1$ for $x \in (-1, \infty)$, and $\lim_{x \to \infty} f(x) = 1$, are both trivial. Hence $M_0 = 1$. Also, since $f''(x) = 4$ for $x \in (-1, 0]$, and $\lim_{x \to 0+} f''(x) = 4$, we see that f is twice-differentiable on $(-1, \infty)$. To show that $|f''(x)| < 4$ for all $x > 0$, observe that

$$3x^2 - 1 < 3x^2 + 1 < x^6 + 3x^4 + 3x^2 + 1 = (x^2+1)^3$$

and that $\left|\frac{3x^2-1}{(x^2+1)^3}\right| = \frac{3x^2-1}{(x^2+1)^3} < 1$ for $x > 0$. So

$$|f''(x)| = 4\left|\frac{3x^2-1}{(x^2+1)^3}\right| < 4$$

for $x > 0$, and hence $M_2 = 4$. Finally, since $0 \le f'(x) < 4$ for $x \in (-1, 0]$, and

$$2x \le 1 + x^2 < 1 + 2x^2 + x^4 = (x^2+1)^2$$

for $x > 0$. We see that

$$|f'(x)| = \left|\frac{4x}{(x^2+1)^2}\right| = 2\left[\frac{2x}{(x^2+1)^2}\right] < 2 < 4$$

for $x > 0$, and hence $M_1 = 4$ (since $\lim_{t \to (-1)+} f'(x) = -4$).
(iii) Yes. In R^k, let $\mathbf{f}(x) = (f_1(x), \dots, f_k(x))$ where each f_j ($1 \le j \le k$) is a twice-differentiable real function. Note that $M_0 M_2$ be of the form $0 \cdot \infty$ is exceptional, and we consider the following four cases. (The first three cases are simple extensions of $k = 1$.)
Case 1: $M_0 = \infty$ or $M_2 = \infty$. The result is trivial.
Case 2: $M_0 = 0$. Then $f_j(x) = 0$, and then $f_j'(x) = f_j''(x) = 0$ for all $x \in (a, \infty)$ and for all j. It follows that $M_1 = M_2 = 0$, and the result is also trivial.
Case 3: $0 < M_0 < \infty$ and $M_2 = 0$. Then $f_j''(x) = 0$, and then $f_j'(x) = c_j$ and $f_j(x) = c_j x + d_j$ for some constants $c_j, d_j \in R$, for all $x \in (a, \infty)$ and for all j. Since M_0 is finite, we need $c_j = 0$ for each j, and then $M_1 = 0$. The result therefore follows.
Case 4: $0 < M_0 < \infty$ and $0 < M_2 < \infty$. If $M_1 = 0$ then we are done. If $M_1 > 0$, let $p \in R^1$ be such that $0 < p < M_1$, and let $x_0 \in (a, \infty)$ be such that $|\mathbf{f}'(x_0)| > p$. Put $\mathbf{u} = \frac{\mathbf{f}'(x_0)}{|\mathbf{f}'(x_0)|}$. Consider the real-valued function $g(x) = \mathbf{u} \cdot \mathbf{f}(x)$ for $x \in (a, \infty)$, and note that g is twice-differentiable. Let N_0, N_1, N_2 be the least upper bounds of $|g(x)|$, $|g'(x)|$, $|g''(x)|$, respectively. Since $|\mathbf{u}| = 1$, by Schwarz inequality [Theorem 1.37(d)],

$$|g(x)| \le |\mathbf{u}|\,|\mathbf{f}(x)| = |\mathbf{f}(x)|\,, \quad |g''(x)| \le |\mathbf{u}|\,|\mathbf{f}''(x)| = |\mathbf{f}''(x)|$$

for all $x \in (a, \infty)$. So that $N_0 \le M_0$ and $N_2 \le M_2$. Also, since

$$N_1 \ge g'(x_0) = \mathbf{u} \cdot \mathbf{f}'(x_0) = |\mathbf{f}'(x_0)| > p$$

and since $N_1 \le 4N_0N_2$ [by part (i)], we have $p < 4M_0M_2$. Since p is arbitrarily chosen such that $0 < p < M_1$, we conclude that $M_1 \le 4M_0M_2$.

---

16. Suppose f is twice-differentiable on $(0, \infty)$, f'' is bounded on $(0, \infty)$, and $f(x) \to 0$ as $x \to \infty$. Prove that $f'(x) \to 0$ as $x \to \infty$.
Hint: Let $a \to \infty$ in Exercise 15.
——

Let $M > 0$ be such that $|f''(x)| \le M$ for $x \in (0, \infty)$. Given $a \in (0, \infty)$, let $M_0(a)$, $M_1(a)$, $M_2(a)$ be the least upper bounds of $|f(x)|$, $|f'(x)|$, $|f''(x)|$, respectively, for $x \in (a, \infty)$. Then clearly $M_2(a) \le M$ for all a. Since $f(x) \to 0$ as $x \to \infty$, for every $\varepsilon > 0$ there exists $a_0 \in (0, \infty)$ such that $|f(x)| < \varepsilon$ for all $x \in (a_0, \infty)$. It follows that $M_0(a_0) \le \varepsilon$ and that (by Exercise 15)

$$M_1^2(a_0) \le 4M_0(a_0)M_2(a_0) \le 4M\varepsilon$$

i.e., $|f'(x)| \le 2\sqrt{M\varepsilon}$ for all $x \in (a_0, \infty)$. Hence $f'(x) \to 0$ as $x \to \infty$.

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17. Suppose f is a real, three times differentiable function on $[-1,1]$, such that

$$f(-1)=0, \quad f(0)=0, \quad f(1)=1, \quad f'(0)=0$$

Prove that $f^{(3)}(x)\geq 3$ for some $x\in(-1,1)$.
Note that equality holds for $\frac{1}{2}(x^3+x^2)$.
Hint: Use Theorem 5.15, with $\alpha=0$ and $\beta=\pm1$, to show that there exist $s\in(0,1)$ and $t\in(-1,0)$ such that

$$f^{(3)}(s)+f^{(3)}(t)=6$$

—–

Following the hint, use Taylor's theorem (Theorem 5.15), there exists $s\in(0,1)$ and $t\in(-1,0)$ such that

$$1=f(1)=f(0)+f'(0)+\frac{f''(0)}{2}+\frac{f^{(3)}(s)}{6}=\frac{f''(0)}{2}+\frac{f^{(3)}(s)}{6}$$
$$0=f(-1)=f(0)-f'(0)+\frac{f''(0)}{2}-\frac{f^{(3)}(s)}{6}=\frac{f''(0)}{2}-\frac{f^{(3)}(t)}{6}$$

Subtract them we get

$$f^{(3)}(s)+f^{(3)}(t)=6$$

It follows that either $f^{(3)}(s)\geq 3$ or $f^{(3)}(t)\geq 3$, which completes the proof.

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18. Suppose f is a real function on $[a,b]$, n is a positive integer, and $f^{(n-1)}$ exists for every $t\in[a,b]$. Let α, β, and P be as in Taylor's theorem (5.15). Define

$$Q(t)=\frac{f(t)-f(\beta)}{t-\beta}$$

for $t\in[a,b]$, $t\neq\beta$, differentiate

$$f(t)-f(\beta)=(t-\beta)Q(t)$$

$n-1$ times at $t=\alpha$, and derive the following version of Taylor's theorem:

$$f(\beta)=P(\beta)+\frac{Q^{(n-1)}(\alpha)}{(n-1)!}(\beta-\alpha)^n$$

—–

By induction, we easily conclude that

$$f^{(k)}(\alpha)=kQ^{(k-1)}(\alpha)-(\beta-\alpha)Q^{(k)}(\alpha)$$

and then

$$\frac{f^{(k)}(\alpha)}{k!}(\beta-\alpha)^k = \frac{Q^{(k-1)}(\alpha)}{(k-1)!}(\beta-\alpha)^k - \frac{Q^{(k)}(\alpha)}{k!}(\beta-\alpha)^{k+1}$$

for $k = 1, 2, \ldots, n-1$. It follows that

$$\begin{aligned} P(\beta) &= f(\alpha) + \sum_{k=1}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(\beta-\alpha)^k \\ &= f(\alpha) + Q(\alpha)(\beta-\alpha) - \frac{Q^{(n-1)}(\alpha)}{(n-1)!}(\beta-\alpha)^n \\ &= f(\alpha) + [f(\beta) - f(\alpha)] - \frac{Q^{(n-1)}(\alpha)}{(n-1)!}(\beta-\alpha)^n \\ &= f(\beta) - \frac{Q^{(n-1)}(\alpha)}{(n-1)!}(\beta-\alpha)^n \end{aligned}$$

or $f(\beta) = P(\beta) + \frac{Q^{(n-1)}(\alpha)}{(n-1)!}(\beta-\alpha)^n$.

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19. Suppose f is defined in $(-1,1)$ and $f'(0)$ exists. Suppose $-1 < \alpha_n < \beta_n < 1$, $\alpha_n \to 0$, and $\beta_n \to 0$ as $n \to \infty$. Define the difference quotients

$$D_n = \frac{f(\beta_n) - f(\alpha_n)}{\beta_n - \alpha_n}$$

Prove the following statements:
(a) If $\alpha_n < 0 < \beta_n$, then $\lim D_n = f'(0)$.
(b) If $0 < \alpha_n < \beta_n$ and $\{\beta_n/(\beta_n - \alpha_n)\}$ is bounded, then $\lim D_n = f'(0)$.
(c) If f' is continuous in $(-1,1)$, then $\lim D_n = f'(0)$.
Give an example in which f is differentiable in $(-1,1)$ (but f' is not continuous at 0) and in which α_n, β_n tend to 0 in such a way that $\lim D_n$ exists but is different from $f'(0)$.
——

(a) Denote $\lambda_n = \frac{\beta_n}{\beta_n - \alpha_n}$ for $n = 1, 2, \ldots$, then clearly $0 < \lambda_n < 1$ (since $\alpha_n < 0 < \beta_n$). Now, we can write

$$\begin{aligned} D_n &= \frac{f(\beta_n) - f(0) + f(0) - f(\alpha_n)}{\beta_n - \alpha_n} \\ &= \frac{\beta_n}{\beta_n - \alpha_n} \cdot \frac{f(\beta_n) - f(0)}{\beta_n} + \frac{-\alpha_n}{\beta_n - \alpha_n} \cdot \frac{f(0) - f(\alpha_n)}{-\alpha_n} \\ &= \lambda_n \cdot \frac{f(\beta_n) - f(0)}{\beta_n} + (1-\lambda_n) \cdot \frac{f(\alpha_n) - f(0)}{\alpha_n} \end{aligned}$$

Since $f'(0)$ exists, by definition, for $\varepsilon > 0$ there exists a positive integer N such that $n \geq N$ implies

$$\left|\frac{f(\beta_n)-f(0)}{\beta_n} - f'(0)\right| < \varepsilon \quad \text{and} \quad \left|\frac{f(\alpha_n)-f(0)}{\alpha_n} - f'(0)\right| < \varepsilon$$

So $n \geq N$ implies

$$\begin{aligned}
\left|D_n - f'(0)\right| &= \left|\lambda_n \left[\frac{f(\beta_n)-f(0)}{\beta_n} - f'(0)\right] + (1-\lambda_n)\left[\frac{f(\alpha_n)-f(0)}{\alpha_n} - f'(0)\right]\right| \\
&\leq \lambda_n \left|\frac{f(\beta_n)-f(0)}{\beta_n} - f'(0)\right| + (1-\lambda_n)\left|\frac{f(\alpha_n)-f(0)}{\alpha_n} - f'(0)\right| \\
&< \lambda_n\varepsilon + (1-\lambda_n)\,\varepsilon \\
&= \varepsilon
\end{aligned}$$

Hence $\lim_{n\to\infty} D_n = f'(0)$.

—–

(b) Let λ_n be the same notation as in part (a), then we know that $\lambda_n > 0$ for each n. Since $\{\beta_n/(\beta_n - \alpha_n)\}$ is bounded, there exists $M > 0$ such that $\lambda_n \leq M$ for each n. Now, we can write

$$\begin{aligned}
D_n &= \frac{[f(\beta_n)-f(0)] - [f(\alpha_n)-f(0)]}{\beta_n - \alpha_n} \\
&= \frac{\beta_n}{\beta_n - \alpha_n}\cdot\frac{f(\beta_n)-f(0)}{\beta_n} + \frac{-\alpha_n}{\beta_n-\alpha_n}\cdot\frac{f(\alpha_n)-f(0)}{\alpha_n} \\
&= \lambda_n\cdot\frac{f(\beta_n)-f(0)}{\beta_n} + (1-\lambda_n)\cdot\frac{f(\alpha_n)-f(0)}{\alpha_n}
\end{aligned}$$

Then if we let ε and N be the same meaning as in part (a), it follows that $n \geq N$ implies

$$\begin{aligned}
\left|D_n - f'(0)\right| &= \left|\lambda_n \left[\frac{f(\beta_n)-f(0)}{\beta_n} - f'(0)\right] + (1-\lambda_n)\left[\frac{f(\alpha_n)-f(0)}{\alpha_n} - f'(0)\right]\right| \\
&\leq \lambda_n \left|\frac{f(\beta_n)-f(0)}{\beta_n} - f'(0)\right| + (1+\lambda_n)\left|\frac{f(\alpha_n)-f(0)}{\alpha_n} - f'(0)\right| \\
&< \lambda_n\varepsilon + (1+\lambda_n)\,\varepsilon \\
&\leq (1+2M)\,\varepsilon
\end{aligned}$$

Hence $\lim_{n\to\infty} D_n = f'(0)$.

—–

(c) By the mean value theorem, for each n there exists $\gamma_n \in (\alpha_n, \beta_n)$ such that

$$D_n = f'(\gamma_n)$$

By squeezing, we see that $\gamma_n \to 0$ as $n \to \infty$. Since f' is continuous, $D_n = f'(\gamma_n) \to f'(0)$ as $n \to \infty$.

−−
Inspired from Exercise 13, we give an example as below. Let $f:(-1,1)\to R^1$ be defined by

$$f(x)=\begin{cases}0 & x=0\\ x^2\sin\left(\frac{1}{x}\right) & \text{otherwise}\end{cases}$$

Then $f'(x)=2x\sin\left(\frac{1}{x}\right)-\cos\left(\frac{1}{x}\right)$ for $x\neq 0$, and

$$f'(0)=\lim_{x\to 0}\frac{f(x)-f(0)}{x}=\lim_{x\to 0}x\sin\left(\frac{1}{x}\right)=0$$

i.e., such f is differentiable on $(-1,1)$ but f' is not continuous at 0. Moreover, let $\alpha_n=\left(2n\pi+\frac{\pi}{2}\right)^{-1}$ and $\beta_n=(2n\pi)^{-1}$ for $n=1,2,\ldots$. Then $0<\alpha_n<\beta_n<1$, $\alpha_n\to 0$, and $\beta_n\to 0$ as $n\to\infty$. But

$$\begin{aligned}D_n&=\frac{-\left(2n\pi+\frac{\pi}{2}\right)^{-2}}{(2n\pi)^{-1}-\left(2n\pi+\frac{\pi}{2}\right)^{-1}}\\&=\frac{-1}{\left(2n\pi+\frac{\pi}{2}\right)^2(2n\pi)^{-1}-\left(2n\pi+\frac{\pi}{2}\right)}\\&=\frac{-1}{\left(2n\pi+\frac{\pi}{2}\right)\left(1+\frac{1}{4n}\right)-\left(2n\pi+\frac{\pi}{2}\right)}\\&=\frac{-4n}{2n\pi+\frac{\pi}{2}}\end{aligned}$$

which follows that $\lim_{n\to\infty}D_n=-\frac{2}{\pi}\neq 0=f'(0)$.

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20. Formulate and prove an inequality which follows form Taylor's theorem and which remains valid for vector-valued functions.

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(i) The statement in the real-valued case: [i]Suppose f is a real function on $[a,b]$, n is a positive integer, $f^{(n-1)}$ is continuous on $[a,b]$, $f^{(n)}(t)$ exists for every $t\in(a.b)$. Let

$$P(t)=\sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{k!}(t-a)^k$$

then there exists $x\in(a,b)$ such that[/i]

$$|f(b)-P(b)|\leq\left|\frac{f^{(n)}(x)}{n!}\right|(b-a)^n$$

The proof of the assertion directly follows from the Taylor' theorem.
(ii) The statement in the vector-valued case: [i]Suppose $\mathbf{f}$ is a mapping on $[a,b]$ into R^k, n is a positive integer, $\mathbf{f}^{(n-1)}$ is continuous on $[a,b]$, $\mathbf{f}^{(n)}(t)$ exists for every $t \in (a.b)$. Let

$$\mathbf{P}(t) = \sum_{k=0}^{n-1} \frac{\mathbf{f}^{(k)}(a)}{k!}(t-a)^k$$

then there exists $x \in (a,b)$ such that[/i]

$$|\mathbf{f}(b) - \mathbf{P}(b)| \leq \left| \frac{\mathbf{f}^{(n)}(x)}{n!} \right| (b-a)^n$$

To show it, note that $|\mathbf{f}(b) - \mathbf{P}(b)| = 0$ is a trivial case. If $|\mathbf{f}(b) - \mathbf{P}(b)| \neq 0$, define

$$\mathbf{u} = \frac{1}{|\mathbf{f}(b) - \mathbf{P}(b)|} [\mathbf{f}(b) - \mathbf{P}(b)]$$

Then $\mathbf{u} \cdot \mathbf{f}$ is a real-valued function on $[a,b]$, satisfying all the conditions in the statement of part (i). Since $(\mathbf{u} \cdot \mathbf{f})^{(k)} = \mathbf{u} \cdot \mathbf{f}^{(k)}$ for $k = 0, 1, 2, \ldots, n$, and

$$\begin{aligned} \sum_{k=0}^{n-1} \frac{(\mathbf{u} \cdot \mathbf{f})^{(k)}(a)}{k!}(t-a)^k &= \sum_{k=0}^{n-1} \frac{\mathbf{u} \cdot \mathbf{f}^{(k)}(a)}{k!}(t-a)^k \\ &= \mathbf{u} \cdot \left[\sum_{k=0}^{n-1} \frac{\mathbf{f}^{(k)}(a)}{k!}(t-a)^k \right] \\ &= \mathbf{u} \cdot \mathbf{P}(t) \end{aligned}$$

by part (i) we have

$$|\mathbf{u} \cdot \mathbf{f}(b) - \mathbf{u} \cdot \mathbf{P}(b)| \leq \left| \frac{\mathbf{u} \cdot \mathbf{f}^{(n)}(x)}{n!} \right| (b-a)^n \leq \left| \frac{\mathbf{f}^{(n)}(x)}{n!} \right| (b-a)^n$$

for some $x \in (a,b)$, where the above second inequality follows by the fact $|\mathbf{u}| = 1$, and by the Schwarz inequality. On the other hand,

$$\mathbf{u} \cdot \mathbf{f}(b) - \mathbf{u} \cdot \mathbf{P}(b) = \mathbf{u} \cdot [\mathbf{f}(b) - \mathbf{P}(b)] = |\mathbf{f}(b) - \mathbf{P}(b)|$$

we then conclude that

$$|\mathbf{f}(b) - \mathbf{P}(b)| \leq \left| \frac{\mathbf{f}^{(n)}(x)}{n!} \right| (b-a)^n$$

21 Let E be a closed subset of R^1. We saw in Exercise 22, Chap. 4, that there is a real continuous function f on R^1 whose zero set is E. It is possible, for each closed set E, to find such an f which is differentiable on R^1, or one which is n times differentiable, or even one which has derivatives of all orders on R^1?

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22. Suppose f is a real function on $(-\infty, \infty)$. Call x a *fixed point* of f if $f(x) = x$.
(a) If f is differentiable and $f'(t) \neq 1$ for every real t, prove that f has at most one fixed point.
(b) Show that the function f defined by

$$f(t) = t + (1 + e^t)^{-1}$$

has no fixed point, although $0 < f'(t) < 1$ for all real t.
(c) However, if there is a constant $A < 1$ such that $|f'(t)| \leq A$ for all real t, prove that a fixed point x of f exists, and that $x = \lim x_n$, where x_1 is an arbitrary real number and

$$x_{n+1} = f(x_n)$$

for $x = 1, 2, 3, \ldots$.
(d) Show that the process described in (c) can be visualized by the zig-zag path

$$(x_1, x_2) \to (x_2, x_2) \to (x_2, x_3) \to (x_3, x_3) \to (x_3, x_4) \to \cdots$$

—–

(a) Suppose there are two fixed points of f, saying x and y. W.l.o.g., let $x < y$. Since f is differentiable, by the mean value theorem, there exists a point $t \in (x, y)$ such that $f(y) - f(x) = (y - x)f'(t)$. But since $f(x) = x$ and $f(y) = y$, we have

$$y - x = (y - x)f'(t)$$

implying $f'(t) = 1$.
—–
(b) Note that $f(t) = t$ if and only if $(1 + e^t)^{-1} = 0$, which is impossible. So f has no fixed point. Next, observe that

$$f'(t) = 1 - \frac{e^t}{(1 + e^t)^2} \in (0, 1)$$

since $\frac{e^t}{(1+e^t)^2} \in (0, 1)$.
—–
(c) The uniqueness of the fixed point x of f follows by part (a). To show the existence, given $x_1 \in R^1$ and define

$$x_{n+1} = f(x_n)$$

for $n = 1, 2, \ldots$. Then by induction and by the mean value theorem,

$$\begin{aligned} |x_{n+1} - x_n| &= |f(x_n) - f(x_{n-1})| \\ &= \left| f'(c_n)(x_n - x_{n-1}) \right| \\ &\le A\,|x_n - x_{n-1}| \\ &\le A^2\,|x_{n-1} - x_{n-2}| \\ &\vdots \\ &\le A^{n-1}\,|x_2 - x_1| \end{aligned}$$

Since $0 < A < 1$, for every $\varepsilon > 0$ there exists a positive N such that $\frac{A^N}{1-A}|x_2 - x_1| < \varepsilon$. It follows that for integers m, n with $n \ge m \ge N + 1$,

$$\begin{aligned} |x_n - x_m| &\le |x_{m+1} - x_m| + |x_{m+2} - x_{m+1}| + \cdots + |x_n - x_{n-1}| \\ &\le \left(A^{m-1} + A^m + \cdots + A^{n-2}\right)|x_2 - x_1| \\ &\le \frac{A^{m-1}}{1-A}|x_2 - x_1| \\ &\le \frac{A^N}{1-A}|x_2 - x_1| \\ &< \varepsilon \end{aligned}$$

i.e., $\{x_n\}$ forms a Cauchy sequence in R^1, and then $\lim_{n\to\infty} x_n = x$ for some $x \in R^1$. Note that such x is the desired fixed point of f, since

$$f(x) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_{n+1} = x$$

—–
(d) The visualization is clear.

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23. The function f defined by

$$f(x) = \frac{x^3 + 1}{3}$$

has three fixed points, say α, β, γ, where

$$-2 < \alpha < -1, \quad 0 < \beta < 1, \quad 1 < \gamma < 2$$

For arbitrarily chosen x_1, define $\{x_n\}$ by setting $x_{n+1} = f(x_n)$.
(a) If $x_1 < \alpha$, prove that $x_n \to -\infty$ as $n \to \infty$.
(b) If $\alpha < x_1 < \gamma$, prove that $x_n \to \beta$ as $n \to \infty$.
(c) If $\gamma < x_1$, prove that $x_n \to +\infty$ as $n \to \infty$.
Thus β can be located by this method, but α and γ cannot.
—–

Define $g(x) = f(x) - x$, then $g(\alpha) = g(\beta) = g(\gamma) = 0$ and

$$g(-2) = -\frac{1}{3}, \quad g(-1) = 1, \quad g(0) = \frac{1}{3}, \quad g(1) = -\frac{1}{3}, \quad g(2) = 1$$

We claim that
(1) $g(x) < 0$ for $x < \alpha$;
(2) $g(x) > 0$ for $\alpha < x < \beta$;
(3) $g(x) < 0$ for $\beta < x < \gamma$;
(4) $g(x) > 0$ for $\gamma < x$.
Proof of the claim. We only prove (1) because the others are in a similar way. Since g is a polynomial of degree 3 and g has three zeros α, β, γ, it is impossible that $g(x) = 0$ whenever $x < \alpha$. Now, suppose there exists $x_0 < \alpha$ such that $g(x_0) > 0$, then $g(-2) < 0 < g(x_0)$ implies there exists c between -2 and x_0, such that $g(c) = 0$. Since $-2 < \alpha$ and $x_0 < \alpha$, we have $c < \alpha$, which is impossible to happen. □
To complete the proof, we consider the following five cases.
Case 1: $x_1 < \alpha$. Suppose $x_n < \alpha$, then $x_{n+1} = \frac{x_n^3+1}{3} < \frac{\alpha^3+1}{3} = \alpha$ (since x^3 monotonically increases). So by mathematical induction, $x_n < \alpha$ for all n. It follows from (1) that $x_{n+1} = g(x_n) + x_n < x_n$ for each n, i.e., $\{x_n\}$ is monotonically decreasing. Suppose $\{x_n\}$ is bounded below, then $\lim_{n\to\infty} x_n = x'$ for some $x' \in R^1$, and

$$f(x') = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_{n+1} = x'$$

i.e., x' is a fixed point of f. Since $g(x') = 0$ and $x' \le x_1 < \alpha$, a contradiction occurs (since g has only three zeros). Hence $\lim_{n\to\infty} x_n = -\infty$.
Case 2: $\alpha < x_1 < \beta$. Suppose $\alpha < x_n < \beta$, then $x_{n+1} = \frac{x_n^3+1}{3} > \frac{\alpha^3+1}{3} = \alpha$ and $x_{n+1} = \frac{x_n^3+1}{3} < \frac{\beta^3+1}{3} = \beta$ (since x^3 monotonically increases). So by mathematical induction, $\alpha < x_n < \beta$ for all n. It follows from (2) that $x_{n+1} = g(x_n) + x_n > x_n$ for each n, i.e., $\{x_n\}$ is monotonically increasing. Since $\{x_n\}$ is bounded above (by β), we have $\lim_{n\to\infty} x_n = x'$ for some $x' \in R^1$, and

$$f(x') = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_{n+1} = x'$$

i.e., x' is a fixed point of f. Since $\alpha < x_1 \le x' \le \beta$, $g(x') = 0$, and $g(x) > 0$ for all $x \in (\alpha, \beta)$, this means that $x' = \beta$.
Case 3: $x_1 = \beta$. It is clear that $x_n = \beta$ for all n, implying $\lim_{n\to\infty} x_n = \beta$.
Case 4: $\beta < x_1 < \gamma$. Similar to Case 2, the conclusion is $\lim_{n\to\infty} x_n = \beta$.
Case 5: $\gamma < x_1$. Similar to Case 1, the conclusion is $\lim_{n\to\infty} x_n = +\infty$.
In summary, part (a) follows by Case 1; part (b) follows by Case 2, 3, and 4; part (c) follows by Case 5.

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24. The process described in part (c) of Exercise 22 can of course also be applied to functions that map $(0, \infty)$ to $(0, \infty)$.

Fix some $\alpha > 1$, and put

$$f(x) = \frac{1}{2}\left(x + \frac{\alpha}{x}\right), \quad g(x) = \frac{\alpha + x}{1 + x}$$

Both f and g have $\sqrt{\alpha}$ as their fixed point in $(0, \infty)$. Try to explain, on the basis of properties of f and g, why the convergence in Exercise 16, Chap. 3, is so much more rapid than it is in Exercise 17. (Compare f' and g', draw the zig-zags suggested in Exercise 22.)
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25. Suppose f is twice differentiable on $[a, b]$, $f(a) < 0$, $f(b) > 0$, $f'(x) \geq \delta > 0$, and $0 \leq f''(x) \leq M$ for all $x \in [a, b]$. Let ξ be the unique point in (a, b) at which $f(\xi) = 0$.
Complete the details in the following outline of *Newton's method* for computing ξ.
(a) Choose $x_1 \in (\xi, b)$, and define $\{x_n\}$ by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Interpret this geometrically, in terms of a tangent to the graph of f.
(b) Prove that $x_{n+1} \leq x_n$ (c.f. Rudin's book says $x_{n+1} < x_n$, but sometimes "$=$" may hold. For example, consider $f(x) = cx + d$ where $c > 0$, then $x_n = \xi = -d/c$ for $n = 2, 3, \ldots$.) and that

$$\lim_{n \to \infty} x_n = \xi$$

(c) Use Taylor's theorem to show that

$$x_{n+1} - \xi = \frac{f''(t_n)}{2f'(x_n)}(x_n - \xi)^2$$

for some $t_n \in (\xi, x_n)$.
(d) If $A = M/2\delta$, deduce that

$$0 \leq x_{n+1} - \xi \leq \frac{1}{A}\left[A(x_1 - \xi)\right]^{2^n}$$

(Compare with Exercise 16 and 18, Chap. 3.)
(e) Show that Newton's method amounts to finding a fixed point of the function g defined by

$$g(x) = x - \frac{f(x)}{f'(x)}$$

How does $g'(x)$ behave for x near ξ?
(f) Put $f(x) = x^{1/3}$ on $(-\infty, \infty)$ and try Newton's method. What happens?
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(a) Note that the tangent line of the graph of f passing through $(x_n, f(x_n))$ is of the form

$$y - f(x_n) = f'(x_n)(x - x_n)$$

Thus $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ is the intersection of the tangent line and the x-axis.
—–
(b) We first show that $x_{n+1} \leq x_n$. Since $f'' \geq 0$, the function f' is increasing. If $x_n = \xi$ for some n, then clearly $x_m = \xi$ for all $m \geq n$ and it is nothing to prove. If $x_n > \xi$, then by the mean value theorem there exists $c_n \in (\xi, x_n)$ such that

$$f(x_n) = f(x_n) - f(\xi) = f'(c_n)(x_n - \xi) \leq f'(x_n)(x_n - \xi)$$

Since $f' \geq \delta > 0$, this reveals that $\frac{f(x_n)}{f'(x_n)} \leq x_n - \xi$, and that

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \geq x_n - (x_n - \xi) = \xi$$

Note that $x_1 > \xi$, so by induction, we conclude that $x_n \geq \xi$ for all n, and then $f(x_n) \geq 0$ for all n (since $f' > 0$ implies f is increasing). It follows that $\frac{f(x_n)}{f'(x_n)} \geq 0$, or equivalently $x_{n+1} \leq x_n$ for all n.
We next show that $\lim_{n\to\infty} x_n = \xi$. Since $\{x_n\}$ is monotonically decreasing with a lower bound ξ, $\lim_{n\to\infty} x_n = x$ for some $x \geq \xi$. But then $x = x - \frac{f(x)}{f'(x)}$, implying $f(x) = 0$, so by the uniqueness, $x = \xi$.
—–
(c) By Taylor's theorem, there exists $t_n \in (\xi, x_n)$ such that

$$f(\xi) = f(x_n) + f'(x_n)(\xi - x_n) + \frac{f''(t_n)}{2}(\xi - x_n)^2$$

Since $f(\xi) = 0$, we have

$$\begin{aligned} x_{n+1} - \xi &= x_n - \xi - \frac{f(x_n)}{f'(x_n)} \\ &= x_n - \xi + \frac{1}{f'(x_n)}\left[f'(x_n)(\xi - x_n) + \frac{f''(t_n)}{2}(\xi - x_n)^2\right] \\ &= \frac{f''(t_n)}{2f'(x_n)}(x_n - \xi)^2 \end{aligned}$$

—–
(d) By part (c), we have

$$\begin{aligned} x_{n+1} - \xi &\leq A(x_n - \xi)^2 \\ &\leq A \cdot A^2 (x_{n-1} - \xi)^4 \\ &\vdots \\ &\leq A \cdot A^2 \cdots A^{2^{n-1}} (x_1 - \xi)^{2^n} \\ &= \frac{1}{A}\left[A(x_1 - \xi)\right]^{2^n} \end{aligned}$$

—–
(e) Since $g(x) = x$ if and only if $f(x) = 0$, the result then follows. Next, compute

$$g'(x) = 1 - \frac{[f'(x)]^2 - f(x)f''(x)}{[f'(x)]^2} = \frac{f(x)f''(x)}{[f'(x)]^2}$$

It follows that $g(x)$ tends to 0 as x near ξ.
—–
(f) Given $x_n \in (-\infty, \infty)$ with $x_n \neq 0$, then $f'(x_n) = \frac{1}{3}x_n^{-\frac{2}{3}}$. So

$$x_{n+1} = x_n - \frac{x_n^{\frac{1}{3}}}{\frac{1}{3}x_n^{-\frac{2}{3}}} = -2x_n$$

and we see that $\{x_n\}$ oscillates and diverges.

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26. Suppose f is differentiable on $[a, b]$, $f(a) = 0$, and there is a real number A such that $|f'(x)| \leq A|f(x)|$ on $[a, b]$. Prove that $f(x) = 0$ for all $x \in [a, b]$. *Hint:* Fix $x_0 \in [a, b]$, let

$$M_0 = \sup|f(x)|, \quad M_1 = \sup|f'(x)|$$

for $a \leq x \leq x_0$. For any such x,

$$|f(x)| \leq M_1(x_0 - a) \leq A(x_0 - a)M_0$$

Hence $M_0 = 0$ if $A(x_0 - a) < 1$. That is, $f = 0$ on $[a, b]$. Proceed.
—–

Following the hint, choose some $x_0 \in (a, b]$ such that $A(x_0 - a) < 1$. If $M_0 = 0$, then clearly $f = 0$, and we can proceed. If $M_0 > 0$, note that

$$|f'(x)| \leq A|f(x)| \leq AM_0$$

for $x \in [a, b]$, so $M_1 \leq AM_0$. Now, denote $\delta = 1 - A(x_0 - a)$, then for $x \in [a, x_0]$,

$$|f(x)| = |f'(c)|(x - a) \leq M_1(x_0 - a) \leq A(x_0 - a)M_0 = M_0 - \delta M_0$$

where $c \in (a, x)$ exists because of the mean value theorem. It follows that $M_0 - \delta M_0$ is a new upper bound of $|f(x)|$ for $x \in [a, x_0]$, contradicting the definition of M_0. So $M_0 > 0$ is impossible, and we conclude that $f = 0$ on $[a, x_0]$. Next, choose $x_k = (k+1)x_0 - ka$ for $k = 0, 1, 2, \ldots, n-1$, where $n \geq 1$ satisfying $x_n = b$ and $x_n - x_{n-1} \leq x_0 - a$. By using the same argument, $f = 0$ on $[x_{k-1}, x_k]$, for $k = 1, 2, \ldots, n$. Hence $f = 0$ on $[a, b]$.

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27. Let ϕ be a real function defined on a rectangle R in the plane, given by $a \le x \le b$, $\alpha \le y \le \beta$. A *solution* of the initial-value problem

$$y' = \phi(x, y), \quad y(a) = c \quad (\alpha \le c \le \beta)$$

is, by definition, a differentiable function f on $[a, b]$ such that $f(a) = c$, $\alpha \le f(x) \le \beta$, and

$$f'(x) = \phi(x, f(x)) \quad (a \le x \le b)$$

Prove that such a problem has at most one solution if there is a constant A such that

$$|\phi(x, y_2) - \phi(x, y_1)| \le A\,|y_2 - y_1|$$

whenever $(x, y_1) \in R$ and $(x, y_2) \in R$.
Hint: Apply Exercise 26 to the difference of two solutions. Note that this uniqueness theorem does not hold for the initial-value problem

$$y' = y^{1/2}, \quad y(0) = 0$$

which has two solutions: $f(x) = 0$ and $f(x) = x^2/4$. Find all other solutions.
––

(i) Let f_1 and f_2 be two solutions of the initial-value problem, define the difference function $f = f_2 - f_1$ on $[a, b]$. It suffices to show that $f = 0$ on $[a, b]$. Observe that $f(a) = f_2(a) - f_1(a) = c - c = 0$, and that

$$\begin{aligned} |f'(x)| &= |f_2'(x) - f_1'(x)| \\ &= |\phi(x, f_2(x)) - \phi(x, f_1(x))| \\ &\le A\,|f_2(x) - f_1(x)| \\ &= A\,|f(x)| \end{aligned}$$

for all $x \in [a, b]$. Thus by Exercise 26, $f = 0$ on $[a, b]$.
(ii) It is easy to check that $f(x) = 0$ and $f(x) = \frac{x^2}{4}$ are solutions of the given initial-value problem. To find the others, observe that if f is a *nonzero* solution, then $y' = y^{\frac{1}{2}}$ implies $f'(x) = [f(x)]^{\frac{1}{2}}$. Differentiate it we get

$$f''(x) = \frac{1}{2}\,[f(x)]^{-\frac{1}{2}}\,f'(x) = \frac{1}{2}\,[f(x)]^{-\frac{1}{2}}\,[f(x)]^{\frac{1}{2}} = \frac{1}{2}$$

So $f'(x) = \frac{x}{2} + c$ for some constant c, and then $f(x) = \left(\frac{x}{2} + c\right)^2$. Since $y(0) = 0$ implies $f(0) = 0$, we get $c = 0$. Hence $f(x) = \frac{x^2}{4}$.

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28. Formulate and prove an analogous uniqueness theorem for systems of differential equations of the form

$$y_j' = \phi_j(x, y_1, \ldots, y_k), \quad y_j(a) = c_j, \quad (j = 1, \ldots, k)$$

Note that this can be rewritten in the form

$$\mathbf{y}' = \Phi(x, \mathbf{y}), \quad \mathbf{y}(a) = \mathbf{c}$$

where $\mathbf{y} = (y_1, \ldots, y_k)$ ranges over a k-cell, Φ is the mapping of a $(k+1)$-cell into the Euclidean k-space whose components are the functions $\phi_1, \ldots, \phi_k$, and $\mathbf{c}$ is the vector $(c_1, \ldots, c_k)$. Use Exercise 26, for vector-valued functions.
—–

Statement: [i]Let Φ be a vector-valued mapping of a $(k+1)$-cell $C = [a, b] \times I$ into R^k. Suppose there exists a constant A such that

$$|\Phi(x, \mathbf{y}_2) - \Phi(x, \mathbf{y}_1)| \leq A\,|\mathbf{y}_2 - \mathbf{y}_1|$$

whenever $(x, \mathbf{y}_1) \in C$ and $(x, \mathbf{y}_2) \in C$. Then the initial-value problem

$$\mathbf{y}' = \Phi(x, \mathbf{y}), \quad \mathbf{y}(a) = \mathbf{c} \quad (\mathbf{c} \in I)$$

has at most one solution.[/i]
To show this, let $\mathbf{f}_1$ and $\mathbf{f}_2$ be two solution of the given initial-value problem, define $\mathbf{f} = \mathbf{f}_2 - \mathbf{f}_1$ on $[a, b]$. It suffices to show that $\mathbf{f} = \mathbf{0}$ on $[a, b]$. Observe that $\mathbf{f}(a) = \mathbf{f}_2(a) - \mathbf{f}_1(a) = \mathbf{c} - \mathbf{c} = \mathbf{0}$, and that

$$\begin{aligned}\left|\mathbf{f}'(x)\right| &= \left|\mathbf{f}_2'(x) - \mathbf{f}_1'(x)\right| \\ &= |\Phi(x, \mathbf{f}_2(x)) - \Phi(x, \mathbf{f}_1(x))| \\ &\leq A\,|\mathbf{f}_2(x) - \mathbf{f}_1(x)| \\ &= A\,|\mathbf{f}(x)|\end{aligned}$$

for all $x \in [a, b]$. Thus by Exercise 26 for vector-valued functions, $\mathbf{f} = \mathbf{0}$ on $[a, b]$.

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29. Specialize Exercise 28 by considering the system

$$\begin{aligned} y_j' &= y_{j+1} \quad (j = 1, \ldots, k-1) \\ y_k' &= f(x) - \sum_{j=1}^{k} g_j(x) y_j \end{aligned}$$

where $f, g_1, \ldots, g_k$ are continuous real functions on $[a, b]$, and derive a uniqueness theorem for solutions of the equation

$$y^{(k)} + g_k(x) y^{(k-1)} + \cdots + g_2(x) y' + g_1(x) y = f(x)$$

subject to initial conditions

$$y(a) = c_1, \quad y'(a) = c_2, \quad \ldots, \quad y^{(k-1)}(a) = c_k$$

—–

Put $\Phi(x, y_1, y_2, \ldots, y_k) = \left(y_2, y_3, \ldots, y_k, f(x) - \sum_{j=1}^{k} g_j(x) y_j\right)$ and $\mathbf{c} = (c_1, c_2, \ldots, c_k)$, then the given system coincides with the initial-value problem

$$\mathbf{y}' = \Phi(x, \mathbf{y}), \quad \mathbf{y}(a) = \mathbf{c}$$

Observe that for $\mathbf{y}_1 = (y_{11}, y_{21}, \ldots, y_{k1})$ and $\mathbf{y}_2 = (y_{12}, y_{22}, \ldots, y_{k2})$,

$$|\Phi(x, \mathbf{y}_2) - \Phi(x, \mathbf{y}_1)|^2 = \sum_{i=2}^{k} (y_{i2} - y_{i1})^2 + \left[\sum_{j=1}^{k} g_j(x)(y_{j2} - y_{j1})\right]^2$$

Denote $M = \sup\{|g_j(x)| : x \in [a,b],\ 1 \le j \le k\}$, then

$$\begin{aligned} |\Phi(x, \mathbf{y}_2) - \Phi(x, \mathbf{y}_1)|^2 &\le \sum_{i=2}^{k} (y_{i2} - y_{i1})^2 + M^2 \left[\sum_{j=1}^{k} (y_{j2} - y_{j1})\right]^2 \\ &\le \sum_{i=2}^{k} (y_{i2} - y_{i1})^2 + kM^2 \sum_{j=1}^{k} (y_{j2} - y_{j1})^2 \\ &\le \left(1 + kM^2\right) \sum_{j=1}^{k} (y_{j2} - y_{j1})^2 \\ &= \left(1 + kM^2\right) |\mathbf{y}_2 - \mathbf{y}_1|^2 \end{aligned}$$

Hence the uniqueness of the solution follows.