ITest 2007

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by djmathman

– Multiple Choice

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1 A twin prime pair is a pair of primes (p, q) such that $q = p + 2$. The Twin Prime Conjecture states that there are infinitely many twin prime pairs. What is the arithmetic mean of the two primes in the smallest twin prime pair? (1 is not a prime.)

(A) 4

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2 Find the value of $a + b$ given that (a, b) is a solution to the system

$$3a + 7b = 1977,$$
$$5a + b = 2007.$$

(A) 488 **(B)** 498

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3 An *abundant number* is a natural number, the sum of whose proper divisors is greater than the number itself. For instance, 12 is an abundant number:

$$1 + 2 + 3 + 4 + 6 = 16 > 12.$$

However, 8 is not an abundant number:

$$1 + 2 + 4 = 7 < 8.$$

Which one of the following natural numbers is an abundant number?

(A) 14 **(B)** 28 **(C)** 56

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4 Star flips a quarter four times. Find the probability that the quarter lands heads exactly twice.

(A) $\frac{1}{8}$ **(B)** $\frac{3}{16}$ **(C)** $\frac{3}{8}$

(D) $\frac{1}{2}$

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5 Compute the sum of all twenty-one terms of the geometric series

$$1 + 2 + 4 + 8 + \cdots + 1048576.$$

(A) 2097149 **(B)** 2097151 **(C)** 2097153

(D) 2097157 **(E)** 2097161

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6 Find the units digit of the sum

$$(1!)^2 + (2!)^2 + (3!)^2 + (4!)^2 + \cdots + (2007!)^2.$$

(A) 0 **(B)** 1 **(C)** 3
(D) 5 **(E)** 7 **(F)** 9

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7 An equilateral triangle with side length 1 has the same area as a square with side length s. Find s.

(A) $\frac{\sqrt[4]{3}}{2}$ **(B)** $\frac{\sqrt[4]{3}}{\sqrt{2}}$ **(C)** 1
(D) $\frac{3}{4}$ **(E)** $\frac{4}{3}$ **(F)** $\sqrt{3}$
(G) $\frac{\sqrt{6}}{2}$

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8 Joe is right at the middle of a train tunnel and he realizes that a train is coming. The train travels at a speed of 50 miles per hour, and Joe can run at a speed of 10 miles per hour. Joe hears the train whistle when the train is a half mile from the point where it will enter the tunnel. At that point in time, Joe can run toward the train and just exit the tunnel as the train meets him. Instead, Joe runs away from the train when he hears the whistle. How many seconds does he have to spare (before the train is upon him) when he gets to the tunnel entrance?

(A) 7.2 **(B)** 14.4 **(C)** 36
(D) 10 **(E)** 12 **(F)** 2.4
(G) 25.2 **(H)** 123456789

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9 Suppose that m and n are positive integers such that $m < n$, the geometric mean of m and n is greater than 2007, and the arithmetic mean of m and n is less than 2007. How many pairs (m, n) satisfy these conditions?

(A) 0 **(B)** 1 **(C)** 2
(D) 3 **(E)** 4 **(F)** 5
(G) 6 **(H)** 7 **(I)** 2007

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10 My grandparents are Arthur, Bertha, Christoph, and Dolores. My oldest grandparent is only 4 years older than my youngest grandparent. Each grandfather is two years older than his wife. If Bertha is younger than Dolores, what is the difference between Bertha's age and the mean of my grandparents' ages?

(A) 0 **(B)** 1 **(C)** 2

(D) 3 **(E)** 4 **(F)** 5
(G) 6 **(H)** 7 **(I)** 8
(J) 2007

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11 Consider the "tower of power" $2^{2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}}$, where there are 2007 twos including the base. What is the last (units) digit of this number?

(A) 0 **(B)** 1 **(C)** 2
(D) 3 **(E)** 4 **(F)** 5
(G) 6 **(H)** 7 **(I)** 8
(J) 9 **(K)** 2007

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12 My frisbee group often calls "best of five" to finish our games when it's getting dark, since we don't keep score. The game ends after one of the two teams scores three points (total, not necessarily consecutive). If every possible sequence of scores is equally likely, what is the expected score of the losing team?

(A) $2/3$ **(B)** 1 **(C)** $3/2$
(D) $8/5$ **(E)** $5/8$ **(F)** 2
(G) 0 **(H)** $5/2$ **(I)** $2/5$
(J) $3/4$ **(K)** $4/3$ **(L)** 2007

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13 What is the smallest positive integer k such that the number $\binom{2k}{k}$ ends in two zeros?

(A) 3 **(B)** 4 **(C)** 5
(D) 6 **(E)** 7 **(F)** 8
(G) 9 **(H)** 10 **(I)** 11
(J) 12 **(K)** 13 **(L)** 14
(M) 2007

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14 Let $\phi(n)$ be the number of positive integers $k < n$ which are relatively prime to n. For how many distinct values of n is $\phi(n)$ equal to 12?

(A) 0 **(B)** 1 **(C)** 2
(D) 3 **(E)** 4 **(F)** 5
(G) 6 **(H)** 7 **(I)** 8

(J) 9 **(K)** 10 **(L)** 11
(M) 12 **(N)** 13

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15 Form a pentagon by taking a square of side length 1 and an equilateral triangle of side length 1 and placing the triangle so that one of its sides coincides with a side of the square. Then "circumscribe" a circle around the pentagon, passing through three of its vertices, so that the circle passes through exactly one vertex of the equilateral triangle, and exactly two vertices of the square. What is the radius of the circle?

(A) $\frac{2}{3}$ **(B)** $\frac{3}{4}$ **(C)** 1
(D) $\frac{5}{4}$ **(E)** $\frac{4}{3}$ **(F)** $\frac{\sqrt{2}}{2}$
(G) $\frac{\sqrt{3}}{2}$ **(H)** $\sqrt{2}$ **(I)** $\sqrt{3}$
(J) $\frac{1+\sqrt{3}}{2}$ **(K)** $\frac{2+\sqrt{6}}{2}$ **(L)** $\frac{7}{6}$
(M) $\frac{2+\sqrt{6}}{4}$ **(N)** $\frac{4}{5}$ **(O)** 2007

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16 How many lattice points lie within or on the border of the circle defined in the xy-plane by the equation $x^2+y^2=100$?

(A) 1 **(B)** 2 **(C)** 4
(D) 5 **(E)** 41 **(F)** 42
(G) 69 **(H)** 76 **(I)** 130
(J) 133 **(K)** 233 **(L)** 311
(M) 317 **(N)** 420 **(O)** 520
(P) 2007

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17 If x and y are acute angles such that $x+y=\pi/4$ and $\tan y=1/6$, find the value of $\tan x$.

(A) $\frac{27\sqrt{2}-18}{71}$ **(B)** $\frac{35\sqrt{2}-6}{71}$ **(C)** $\frac{35\sqrt{3}+12}{33}$
(D) $\frac{37\sqrt{3}+24}{33}$ **(E)** 1 **(F)** $\frac{5}{7}$
(G) $\frac{3}{7}$ **(H)** 6 **(I)** $\frac{1}{6}$

(J) $\frac{1}{2}$ **(K)** $\frac{6}{7}$ **(L)** $\frac{4}{7}$

(M) $\sqrt{3}$ **(N)** $\frac{\sqrt{3}}{3}$ **(O)** $\frac{5}{6}$

(P) $\frac{2}{3}$ **(Q)** $\frac{1}{2007}$

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18 Suppose that x^3+px^2+qx+r is a cubic with a double root at a and another root at b, where a and b are real numbers. If $p=-6$ and $q=9$, what is r?

(A) 0 **(B)** 4

(C) 108 **(D)** It could be 0 or 4.

(E) It could be 0 or 108. **(F)** 18

(G) -4 **(H)** -108

(I) It could be 0 or -4. **(J)** It could be 0 or -108.

(K) It could be 4 or -4. **(L)** There is no such value of r.

(M) 1 **(N)** -2

(O) It could be -2 or -4. **(P)** It could be 0 or -2.

(Q) It could be 2007 or a yippy dog. **(R)** 2007

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19 One day Jason finishes his math homework early, and decides to take a jog through his neighborhood. While jogging, Jason trips over a leprechaun. After dusting himself off and apologizing to the odd little magical creature, Jason, thinking there is nothing unusual about the situation, starts jogging again. Immediately the leprechaun calls out, "hey, stupid, this is your only chance to win gold from a leprechaun!"

Jason, while not particularly greedy, recognizes the value of gold. Thinking about his limited college savings, Jason approaches the leprechaun and asks about the opportunity. The leprechaun hands Jason a fair coin and tells him to flip it as many times as it takes to flip a head. For each tail Jason flips, the leprechaun promises one gold coin.

If Jason flips a head right away, he wins nothing. If he first flips a tail, then a head, he wins one gold coin. If he's lucky and flips ten tails before the first head, he wins *ten gold coins*. What is the expected number of gold coins Jason wins at this game?

(A) 0 **(B)** $\frac{1}{10}$ **(C)** $\frac{1}{8}$

(D) $\frac{1}{5}$ **(E)** $\frac{1}{4}$ **(F)** $\frac{1}{3}$

(G) $\frac{2}{5}$ (H) $\frac{1}{2}$ (I) $\frac{3}{5}$
(J) $\frac{2}{3}$ (K) $\frac{4}{5}$ (L) 1
(M) $\frac{5}{4}$ (N) $\frac{4}{3}$ (O) $\frac{3}{2}$
(P) 2 (Q) 3 (R) 4
(S) 2007

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20 Find the largest integer n such that $2007^{1024} - 1$ is divisible by 2^n.

(A) 1 (B) 2 (C) 3
(D) 4 (E) 5 (F) 6
(G) 7 (H) 8 (I) 9
(J) 10 (K) 11 (L) 12
(M) 13 (N) 14 (O) 15
(P) 16 (Q) 55 (R) 63
(S) 64 (T) 2007

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21 James writes down fifteen 1's in a row and randomly writes $+$ or $-$ between each pair of consecutive 1's. One such example is

$$1+1+1-1-1+1-1+1-1+1-1-1-1+1+1.$$

What is the probability that the value of the expression James wrote down is 7?

(A) 0 (B) $\frac{6435}{2^{14}}$ (C) $\frac{6435}{2^{13}}$

(D) $\frac{429}{2^{12}}$ (E) $\frac{429}{2^{11}}$ (F) $\frac{429}{2^{10}}$

(G) $\frac{1}{15}$ (H) $\frac{1}{31}$ (I) $\frac{1}{30}$

(J) $\frac{1}{29}$ (K) $\frac{1001}{2^{15}}$ (L) $\frac{1001}{2^{14}}$

(M) $\frac{1001}{2^{13}}$ (N) $\frac{1}{2^{7}}$ (O) $\frac{1}{2^{14}}$

(P) $\frac{1}{2^{15}}$ (Q) $\frac{2007}{2^{14}}$ (R) $\frac{2007}{2^{15}}$

(S) $\frac{2007}{2^{2007}}$ (T) $\frac{1}{2007}$ (U) $\frac{-2007}{2^{14}}$

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22 Find the value of c such that the system of equations

$$|x+y| = 2007,$$
$$|x-y| = c$$

has exactly two solutions (x, y) in real numbers.

(A) 0 (B) 1 (C) 2

(D) 3 (E) 4 (F) 5

(G) 6 (H) 7 (I) 8

(J) 9 (K) 10 (L) 11

(M) 12 (N) 13 (O) 14

(P) 15 (Q) 16 (R) 17

(S) 18 (T) 223 (U) 678

(V) 2007

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23 Find the product of the non-real roots of the equation

$$(x^2-3)^2+5(x^2-3)+6=0.$$

(A) 0 **(B)** 1 **(C)** -1
(D) 2 **(E)** -2 **(F)** 3
(G) -3 **(H)** 4 **(I)** -4
(J) 5 **(K)** -5 **(L)** 6
(M) -6 **(N)** $3+2i$ **(O)** $3-2i$
(P) $\frac{-3+i\sqrt{3}}{2}$ **(Q)** 8 **(R)** -8
(S) 12 **(T)** -12 **(U)** 42
(V) Ying **(W)** 2007

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24 Let N be the smallest positive integer N such that $2008N$ is a perfect square and $2007N$ is a perfect cube. Find the remainder when N is divided by 25.

(A) 0 **(B)** 1 **(C)** 2
(D) 3 **(E)** 4 **(F)** 5
(G) 6 **(H)** 7 **(I)** 8
(J) 9 **(K)** 10 **(L)** 11
(M) 12 **(N)** 13 **(O)** 14
(P) 15 **(Q)** 16 **(R)** 17
(S) 18 **(T)** 19 **(U)** 20
(V) 21 **(W)** 22 **(X)** 23

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25 Ted's favorite number is equal to

$$1\cdot\binom{2007}{1}+2\cdot\binom{2007}{2}+3\cdot\binom{2007}{3}+\cdots+2007\cdot\binom{2007}{2007}.$$

Find the remainder when Ted's favorite number is divided by 25.

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4 **(F)** 5 **(G)** 6 **(H)** 7 **(I)** 8 **(J)** 9 **(K)** 10 **(L)** 11 **(M)** 12 **(N)** 13 **(O)** 14 **(P)** 15 **(Q)** 16 **(R)** 17 **(S)** 18 **(T)** 19 **(U)** 20 **(V)** 21 **(W)** 22 **(X)** 23 **(Y)** 24

– Short Answer

26 Julie runs a website where she sells university themed clothing. On Monday, she sells thirteen Stanford sweatshirts and nine Harvard sweatshirts for a total of $\$370$. On Tuesday, she sells nine Stanford sweatshirts and two Harvard sweatshirts for a total of $\$180$. On Wednesday, she sells twelve Stanford sweatshirts and six Harvard sweatshirts. If Julie didn't change the prices of any items all week, how much money did she take in (total number of dollars) from the sale of Stanford and Harvard sweatshirts on Wednesday?

27 The face diagonal of a cube has length 4. Find the value of n given that $n\sqrt{2}$ is the *volume* of the cube.

28 The space diagonal (interior diagonal) of a cube has length 6. Find the *surface area* of the cube.

29 Let S be equal to the sum

$$1+2+3+\cdots+2007.$$

Find the remainder when S is divided by 1000.

30 While working with some data for the Iowa City Hospital, James got up to get a drink of water. When he returned, his computer displayed the "blue screen of death" (it had crashed). While rebooting his computer, James remembered that he was nearly done with his calculations since the last time he saved his data. He also kicked himself for not saving before he got up from his desk. He had computed three positive integers a, b, and c, and recalled that their product is 24, but he didn't remember the values of the three integers themselves. What he really needed was their sum. He knows that the sum is an even two-digit integer less than 25 with fewer than 6 divisors. Help James by computing $a + b + c$.

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31 Let x be the length of one side of a triangle and let y be the height to that side. If $x + y = 418$, find the maximum possible *integral value* of the area of the triangle.

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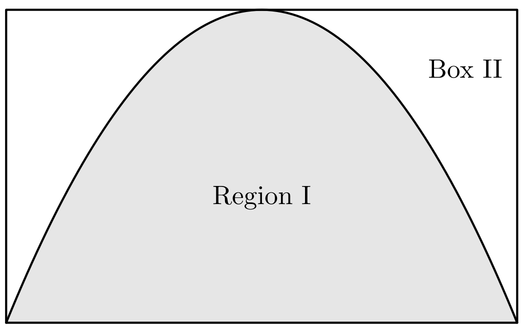
32 When a rectangle frames a parabola such that a side of the rectangle is parallel to the parabola's axis of symmetry, the parabola divides the rectangle into regions whose areas are in the ratio 2 to 1. How many integer values of k are there such that $0 < k \leq 2007$ and the area between the parabola $y = k - x^2$ and the x-axis is an integer?



$$\text{area(I)} = \tfrac{2}{3}\,\text{area(II)}$$

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33 How many *odd* four-digit integers have the property that their digits, read left to right, are in strictly decreasing order?

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34 Let a/b be the probability that a randomly selected divisor of 2007 is a multiple of 3. If a and b are relatively prime positive integers, find $a + b$.

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35 Find the greatest natural number possessing the property that each of its digits except the first and last one is less than the arithmetic mean of the two neighboring digits.

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36 Let b be a real number randomly sepected from the interval $[-17, 17]$. Then, m and n are two relatively prime positive integers such that m/n is the probability that the equation

$$x^4 + 25b^2 = (4b^2 - 10b)x^2$$

has *at least* two distinct real solutions. Find the value of $m + n$.

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37 Rob is helping to build the set for a school play. For one scene, he needs to build a multi-colored tetrahedron out of cloth and bamboo. He begins by fitting three lengths of bamboo together, such that they meet at the same point, and each pair of bamboo rods meet at a right angle. Three more lengths of bamboo are then cut to connect the other ends of the first three rods. Rob then cuts out four triangular pieces of fabric: a blue piece, a red piece, a green piece, and a yellow piece. These triangular pieces of fabric just fill in the triangular spaces between the bamboo, making up the four faces of the tetrahedron. The areas in square feet of the red, yellow, and green pieces are 60, 20, and 15 respectively. If the blue piece is the largest of the four sides, find the number of square feet in its area.

38 Find the largest positive integer that is equal to the cube of the sum of its digits.

39 Let a and b be relatively prime positive integers such that a/b is the sum of the real solutions to the equation
$$\sqrt[3]{3x-4}+\sqrt[3]{5x-6}=\sqrt[3]{x-2}+\sqrt[3]{7x-8}.$$
Find $a+b$.

40 Let S be the sum of all x such that $1\leq x\leq 99$ and
$$\{x^2\}=\{x\}^2.$$
Compute $\lfloor S\rfloor$.

41 The sequence of digits
$$123456789101112131415161718192021\ldots$$
is obtained by writing the positive integers in order. If the 10^nth digit in this sequence occurs in the part of the sequence in which the m-digit numbers are placed, define $f(n)$ to be m. For example, $f(2)=2$ because the 100^{th} digit enters the sequence in the placement of the two-digit integer 55. Find the value of $f(2007)$.

42 During a movie shoot, a stuntman jumps out of a plane and parachutes to safety within a 100 foot by 100 foot square field, which is entirely surrounded by a wooden fence. There is a flag pole in the middle of the square field. Assuming the stuntman is equally likely to land on any point in the field, the probability that he lands closer to the fence than to the flag pole can be written in simplest terms as
$$\dfrac{a-b\sqrt c}d,$$
where all four variables are positive integers, c is a multple of no perfect square greater than 1, a is coprime with d, and b is coprime with d. Find the value of $a+b+c+d$.

43 Bored of working on her computational linguistics thesis, Erin enters some three-digit integers into a spreadsheet, then manipulates the cells a bit until her spreadsheet calculates each of the following 100 9-digit integers:

$$\begin{aligned}&700\cdot 712\cdot 718+320,\\&701\cdot 713\cdot 719+320,\\&702\cdot 714\cdot 720+320,\\&\vdots\\&798\cdot 810\cdot 816+320,\\&799\cdot 811\cdot 817+320.\end{aligned}$$

She notes that two of them have exactly 8 positive divisors each. Find the common prime divisor of those two integers.

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44 A positive integer n between 1 and $N=2007^{2007}$ inclusive is selected at random. If a and b are natural numbers such that a/b is the probability that N and n^3-36n are relatively prime, find the value of $a+b$.

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45 Find the sum of all positive integers B such that $(111)_B=(aabbcc)_6$, where a,b,c represent distinct base 6 digits, $a\neq 0$.

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46 Let (x,y,z) be an ordered triplet of real numbers that satisfies the following system of equations:

$$\begin{aligned}x+y^2+z^4&=0,\\y+z^2+x^4&=0,\\z+x^2+y^4&=0.\end{aligned}$$

If m is the minimum possible value of $\lfloor x^3+y^3+z^3\rfloor$, find the modulo 2007 residue of m.

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47 Let $\{X_n\}$ and $\{Y_n\}$ be sequences defined as follows:

$$X_0=Y_0=X_1=Y_1=1,$$

$$\begin{aligned}X_{n+1}&=X_n+2X_{n-1}\qquad(n=1,2,3\ldots),\\Y_{n+1}&=3Y_n+4Y_{n-1}\qquad(n=1,2,3\ldots).\end{aligned}$$

Let k be the largest integer that satisfies all of the following conditions: - $|X_i-k|\leq 2007$, for some positive integer i;
- $|Y_j-k|\leq 2007$, for some positive integer j; and
- $k<10^{2007}$. Find the remainder when k is divided by 2007.

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48 Let a and b be relatively prime positive integers such that a/b is the maximum possible value of

$$\sin^2 x_1 + \sin^2 x_2 + \sin^2 x_3 + \cdots + \sin^2 x_{2007},$$

where, for $1 \leq i \leq 2007$, x_i is a nonnegative real number, and

$$x_1 + x_2 + x_3 + \cdots + x_{2007} = \pi.$$

Find the value of $a + b$.

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49 How many 7-element subsets of $\{1, 2, 3, \ldots, 14\}$ are there, the sum of whose elements is divisible by 14?

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50 A block Z is formed by gluing one face of a solid cube with side length 6 onto one of the circular faces of a right circular cylinder with radius 10 and height 3 so that the centers of the square and circle coincide. If V is the smallest convex region that contains Z, calculate $\lfloor \operatorname{vol} V \rfloor$ (the greatest integer less than or equal to the volume of V).

– Ultimate Question

-1 The Ultimate Question is a 10-part problem in which each question after the first depends on the answer to the previous problem. As in the Short Answer section, the answer to each (of the 10) problems is a nonnegative integer. You should submit an answer for each of the 10 problems you solve (unlike in previous years). In order to receive credit for the correct answer to a problem, you must also correctly answer *every one of the previous parts of the Ultimate Question*.

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51 Find the highest point (largest possible y-coordinate) on the parabola

$$y = -2x^2 + 28x + 418.$$

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52 Let T = TNFTPP. Let R be the region consisting of the points (x, y) of the cartesian plane satisfying both $|x| - |y| \leq T - 500$ and $|y| \leq T - 500$. Find the area of region R.

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53 Let T = TNFTPP. Three distinct positive Fibonacci numbers, all greater than T, are in arithmetic progression. Let N be the smallest possible value of their sum. Find the remainder when N is divided by 2007.

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54 Let T = TNFTPP. Consider the sequence $(1, 2007)$. Inserting the difference between 1 and 2007 between them, we get the sequence $(1, 2006, 2007)$. Repeating the process of inserting differences between numbers, we get the sequence $(1, 2005, 2006, 1, 2007)$. A third iteration of this process results in $(1, 2004, 2005, 1, 2006, 2005, 1, 2006, 2007)$. A total of 2007 iterations produces a sequence with $2^{2007} + 1$ terms. If the integer $4T$ (that is, 4 times the integer T) appears a total of N times among these $2^{2007} + 1$ terms, find the remainder when N gets divided by 2007.

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55 Let $T=$ TNFTPP, and let $R=T-914$. Let x be the smallest real solution of

$$3x^2+Rx+R=90x\sqrt{x+1}.$$

Find the value of $\lfloor x\rfloor$.

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56 Let $T=$ TNFTPP. In the binary expansion of

$$\frac{2^{2007}-1}{2^T-1},$$

how many of the first $10,000$ digits to the right of the radix point are 0's?

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57 Let $T=$ TNFTPP. How many positive integers are within T of exactly $\lfloor\sqrt{T}\rfloor$ perfect squares? (Note: $0^2=0$ is considered a perfect square.)

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58 Let $T=$ TNFTPP. For natural numbers $k,n\geq 2$, we define $S(k,n)$ such that

$$S(k,n)=\left\lfloor\frac{2^{n+1}+1}{2^{n-1}+1}\right\rfloor+\left\lfloor\frac{3^{n+1}+1}{3^{n-1}+1}\right\rfloor+\cdots+\left\lfloor\frac{k^{n+1}+1}{k^{n-1}+1}\right\rfloor.$$

Compute the value of $S(10,T+55)-S(10,55)+S(10,T-55)$.

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59 Let $T=$ TNFTPP. Fermi and Feynman play the game *Probabicloneme* in which Fermi wins with probability a/b, where a and b are relatively prime positive integers such that $a/b<1/2$. The rest of the time Feynman wins (there are no ties or incomplete games). It takes a negligible amount of time for the two geniuses to play *Probabicloneme* so they play many many times. Assuming they can play infinitely many games (eh, they're in Physicist Heaven, we can bend the rules), the probability that they are ever tied in total wins after they start (they have the same positive win totals) is $(T-332)/(2T-601)$. Find the value of a.

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60 Let $T=$ TNFTPP. Triangle ABC has $AB=6T-3$ and $AC=7T+1$. Point D is on BC so that AD bisects angle BAC. The circle through A, B, and D has center O_1 and intersects line AC again at B', and likewise the circle through A, C, and D has center O_2 and intersects line AB again at C'. If the four points B', C', O_1, and O_2 lie on a circle, find the length of BC.
