ITest 2008

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by djmathman

1 Jerry and Hannah Kubik live in Jupiter Falls with their five children. Jerry works as a Renewable Energy Engineer for the Southern Company, and Hannah runs a lab at Jupiter Falls University where she researches biomass (renewable fuel) conversion rates. Michael is their oldest child, and Wendy their oldest daughter. Tony is the youngest child. Twins Joshua and Alexis are 12 years old.

When the Kubiks went on vacation to San Diego last year, they spent a day at the San Diego Zoo. Single day passes cost $\$33$ for adults (Jerry and Hannah), $\$22$ for children (Michael is still young enough to get the children's rate), and family memberships (which allow the whole family in at once) cost $\$120$. How many dollars did the family save by buying a family pass over buying single day passes for every member of the family?

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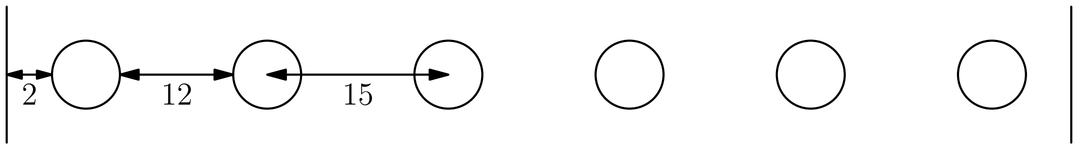
2 One day while Tony plays in the back yard of the Kubik's home, he wonders about the width of the back yard, which is in the shape of a rectangle. A row of trees spans the width of the back of the yard by the fence, and Tony realizes that all the trees have almost exactly the same diameter, and the trees look equally spaced. Tony fetches a tape measure from the garage and measures a distance of almost exactly 12 feet between a consecutive pair of trees. Tony realizes the need to include the width of the trees in his measurements. Unsure as to how to do this, he measures the distance between the centers of the trees, which comes out to be around 15 feet. He then measures 2 feet to either side of the first and last trees in the row before the ends of the yard. Tony uses these measurements to estimate the width of the yard. If there are six trees in the row of trees, what is Tony's estimate in feet?



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3 Michael plays catcher for his school's baseball team. He has always been a great player behind the plate, but this year as a junior, Michael's offense is really improving. His batting average is $.417$ after six games, and the team is $6-0$ (six wins and no losses). They are off to their best start in years.

On the way home from their sixth game, Michael notes to his father that the attendance seems to be increasing due to the team's great start, "There were 181 people at the first game, then 197 at the second, 203 the third, 204 the fourth, 212 at the fifth, and there were 227 at today's game." Just then, Michael's genius younger brother Tony, just seven-years-old, computes the average attendance of the six games. What is their average?

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4 The difference between two prime numbers is 11. Find their sum.

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5 Jerry recently returned from a trip to South America where he helped two old factories reduce pollution output by installing more modern scrubber equipment. Factory A previously filtered 80% of pollutants and Factory B previously filled 72% of pollutants. After installing the new scrubber system, both factories now filter 99.5% of pollutants.

Jerry explains the level of pollution reduction to Michael, "Factory A is the much larger factory. It's four times as large as Factory B. Without any filters at all, it would pollute four times as much as Factory B. Even with the better pollution filtration system, Factory A was polluting nearly three times as much as Factory B."

Assuming the factories are the same in every way except size and previous percentage of pollution filtered, find $a+b$ where a/b is the ratio in lowest terms of volume of pollutants unfiltered from both factories *after* installation of the new scrubber system to the volume of pollutants unfiltered from both factories *before* installation of the new scrubber system.

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6 Let L be the length of the altitude to the hypotenuse of a right triangle with legs 5 and 12. Find the least integer greater than L.

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7 Find the number of integers n for which $n^2 + 10n < 2008$.

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8 The math team at Jupiter Falls Middle School meets twice a month during the Summer, and the math team coach, Mr. Fischer, prepares some Olympics-themed problems for his students. One of the problems Joshua and Alexis work on boils down to a system of equations:

$$\begin{aligned}2x+3y+3z&=8,\\3x+2y+3z&=808,\\3x+3y+2z&=80808.\end{aligned}$$

Their goal is not to find a solution (x, y, z) to the system, but instead to compute the sum of the variables. Find the value of $x + y + z$.

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9 Joshua likes to play with numbers and patterns. Joshua's favorite number is 6 because it is the units digit of his birth year, 1996. Part of the reason Joshua likes the number 6 so much is that the powers of 6 all have the same units digit as they grow from 6^1:

$$\begin{aligned}6^1 &= 6,\\ 6^2 &= 36,\\ 6^3 &= 216,\\ 6^4 &= 1296,\\ 6^5 &= 7776,\\ 6^6 &= 46656,\\ &\vdots\end{aligned}$$

However, not all units digits remain constant when exponentiated in this way. One day Joshua asks Michael if there are simple patterns for the units digits when each one-digit integer is exponentiated in the manner above. Michael responds, ”You tell me!” Joshua gives a disappointed look, but then Michael suggests that Joshua play around with some numbers and see what he can discover. ”See if you can find the units digit of 2008^{2008},” Michael challenges. After a little while, Joshua finds an answer which Michael confirms is correct. What is Joshua's correct answer (the units digit of 2008^{2008})?

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10 Tony has an old sticky toy spider that very slowly ”crawls” down a wall after being stuck to the wall. In fact, left untouched, the toy spider crawls down at a rate of one inch for every two hours it's left stuck to the wall. One morning, at around 9 o' clock, Tony sticks the spider to the wall in the living room three feet above the floor. Over the next few mornings, Tony moves the spider up three feet from the point where he finds it. If the wall in the living room is 18 feet high, after how many days (days after the first day Tony places the spider on the wall) will Tony run out of room to place the spider three feet higher?

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11 After moving his sticky toy spider one morning, Tony heads outside to play ”pirates” with his pal Nick, who lives a few doors down the street from the Kubiks. Tony and Nick imagine themselves as pirates in a rough skirmish over a chest of gold. Victorious over their foes, Tony and Nick claim the prize. However, they must split some of the gold with their crew, which they imagine consists of eight other bloodthirsty pirates. Each of the pirates receives at least one gold coin, but none receive the same number of coins, then Tony and Nick split the remainder equally. If there are 2000 gold coins in the chest, what is the greatest number of gold coins Tony could take as his share? (Assume each gold coin is equally valuable.)

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12 One day while the Kubik family attends one of Michael's baseball games, Tony gets bored and walks to the creek a few yards behind the baseball field. One of Tony's classmates Mitchell sees Tony and goes to join him. While playing around the creek, the two boys find an ordinary six-sided die buried in sediment. Mitchell washes it off in the water and challenges Tony to a contest. Each of the boys rolls the die exactly once. Mitchell's roll is 3 higher than Tony's. ”Let's play once more,” says Tony. Let a/b be the probability that the difference between the outcomes of the two dice is again exactly 3 (regardless of which of the boys rolls higher), where a and b are relatively prime positive integers. Find $a+b$.

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13 In preparation for the family's upcoming vacation, Tony puts together five bags of jelly beans, one bag for each day of the trip, with an equal number of jelly beans in each bag. Tony then pours all the jelly beans out of the five bags and begins making patterns with them. One of the patterns that he makes has one jelly bean in a top row, three jelly beans in the next row, five jelly beans in the row after that, and so on:

$$\begin{array}{ccccccccc} & & & & * & & & & \\ & & & * & * & * & & & \\ & & * & * & * & * & * & & \\ & * & * & * & * & * & * & * & \\ * & * & * & * & * & * & * & * & * \\ & & & & \vdots & & & & \end{array}$$

Continuing in this way, Tony finishes a row with none left over. For instance, if Tony had exactly 25 jelly beans, he could finish the fifth row above with no jelly beans left over. However, when Tony finishes, there are between 10 and 20 rows. Tony then scoops all the jelly beans and puts them all back into the five bags so that each bag once again contains the same number. How many jelly beans are in each bag? (Assume that no marble gets put inside more than one bag.)

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14 The sum of the two perfect cubes that are closest to 500 is $343+512=855$. Find the sum of the two perfect cubes that are closest to 2008.

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15 How many four-digit multiples of 8 are greater than 2008?

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16 In order to encourage the kids to straighten up their closets and the storage shed, Jerry offers his kids some extra spending money for their upcoming vacation. "I don't care what you do, I just want to see everything look clean and organized."

While going through his closet, Joshua finds an old bag of marbles that are either blue or red. The ratio of blue to red marbles in the bag is $17:7$. Alexis also has some marbles of the same colors, but hasn't used them for anything in years. She decides to give Joshua her marbles to put in his marble bag so that all the marbles are in one place. Alexis has twice as many red marbles as blue marbles, and when the twins get all their marbles in one bag, there are exactly as many red marbles and blue marbles, and the total number of marbles is between 200 and 250. How many total marbles do the twins have together?

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17 One day when Wendy is riding her horse Vanessa, they get to a field where some tourists are following Martin (the tour guide) on some horses. Martin and some of the workers at the stables are each leading extra horses, so there are more horses than people. Martin's dog Berry runs around near the trail as well. Wendy counts a total of 28 heads belonging to the people, horses, and dog. She counts a total of 92 legs belonging to everyone, and notes that nobody is missing any legs.

Upon returning home Wendy gives Alexis a little problem solving practice, "I saw 28 heads and 92 legs belonging to people, horses, and dogs. Assuming two legs per person and four for the other animals, how many people did I see?" Alexis scribbles out some algebra and answers correctly. What is her answer?

18 Find the number of lattice points that the line $19x + 20y = 1909$ passes through in Quadrant I.

19 Let A be the set of positive integers that are the product of two consecutive integers. Let B the set of positive integers that are the product of three consecutive integers. Find the sum of the two smallest elements of $A \cap B$.

20 In order to earn a little spending money for the family vacation, Joshua and Wendy offer to clean up the storage shed. After clearing away some trash, Joshua and Wendy set aside give boxes that belong to the two of them that they decide to take up to their bedrooms. Each is in the shape of a cube. The four smaller boxes are all of equal size, and when stacked up, reach the exact height of the large box. If the volume of one of the smaller boxes is 216 cubic inches, find the sum of the volumes of all five boxes (in cubic inches).

21 One of the boxes that Joshua and Wendy unpack has Joshua's collection of board games. Michael, Wendy, Alexis, and Joshua decide to play one of them, a game called *Risk* that involves rolling ordinary six-sided dice to determine the outcomes of strategic battles. Wendy has never played before, so early on Michael explains a bit of strategy.

"You have the first move and you occupy three of the four territories in the Australian continent. You'll want to attack Joshua in Indonesia so that you can claim the Australian continent which will give you bonus armies on your next turn."

"Don't tell her *that!*" complains Joshua.

Wendy and Joshua begin rolling dice to determine the outcome of their struggle over Indonesia. Joshua rolls extremely well, overcoming longshot odds to hold off Wendy's attack. Finally, Wendy is left with one chance. Wendy and Joshua each roll just one six-sided die. Wendy wins if her roll is *higher* than Joshua's roll. Let a and b be relatively prime positive integers so that a/b is the probability that Wendy rolls higher, giving her control over the continent of Australia. Find the value of $a + b$.

22 Tony plays a game in which he takes 40 nickels out of a roll and tosses them one at a time toward his desk where his change jar sits. He awards himself 5 points for each nickel that lands in the jar, and takes away 2 points from his score for each nickel that hits the ground. After Tony is done tossing all 40 nickels, he computes 88 as his score. Find the greatest number of nickels he could have successfully tossed into the jar.

23 Find the number of positive integers n that are solutions to the simultaneous system of inequalities

$$4n - 18 < 2008,$$
$$7n + 17 > 2008.$$

24 In order to earn her vacation spending money, Alexis helped her mother remove weeds from the garden. When she was done, she came into the house to put away her gardening gloves and change into clean clothes.

On her way to her room she notices Joshua with his face to the floor in the family room, looking pretty silly. ”Josh, did you know you lose IQ points for sniffing the carpet?”

”Shut up. I'm *not* sniffing the carpet. I'm *doing something*.”

”Sure, if *sniffing the carpet* counts as *doing something*.” At this point Alexis stands over her twin brother grinning, trying to see how silly she can make him feel.

Joshua climbs to his feet and stands on his toes to make himself a half inch taller than his sister, who is ordinarily a half inch taller than Joshua. ”I'm measuring something. I'm *designing* something.”

Alexis stands on her toes too, reminding her brother that she is still taller than he. ”When you're done, can you design me a dress?”

”Very funny.” Joshua walks to the table and points to some drawings. ”I'm designing the sand castle I want to build at the beach. Everything needs to be measured out so that I can build something awesome.”

”And this requires sniffing carpet?” inquires Alexis, who is just a little intrigued by her brother's project.

”I was imagining where to put the base of a spiral staircase. Everything needs to be measured out correctly. See, the castle walls will be in the shape of a rectangle, like this room. The center of the staircase will be 9 inches from one of the corners, 15 inches from another, 16 inches from another, and some whole number of inches from the furthest corner.” Joshua shoots Alexis a wry smile. The twins liked to challenge each other, and Alexis knew she had to find the distance from the center of the staircase to the fourth corner of the castle on her own, or face Joshua's pestering, which might last for hours or days.

Find the distance from the center of the staircase to the furthest corner of the rectangular castle, assuming all four of the distances to the corners are described as distances on the same plane (the ground).

25 A cube has edges of length 120 cm. The cube gets chopped up into some number of smaller cubes, all of equal size, such that each edge of one of the smaller cubes has an integer length. One of those smaller cubes is then chopped up into some number of *even smaller* cubes, all of equal size. If the edge length of one of those *even smaller* cubes is n cm, where n is an integer, find the number of possible values of n.

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26 Done working on his sand castle design, Joshua sits down and starts rolling a 12-sided die he found when cleaning the storage shed. He rolls and rolls and rolls, and after 17 rolls he finally rolls a 1. Just 3 rolls later he rolls the first 2 *after* that first roll of 1. 11 rolls later, Joshua rolls the first 3 *after* the first 2 that he rolled *after* the first 1 that he rolled. His first 31 rolls make the sequence

$$4, 3, 11, 3, 11, 8, 5, 2, 12, 9, 5, 7, 11, 3, 6, 10, \mathbf{1}, 8, 3, \mathbf{2}, 10, 4, 2, 8, 1, 9, 7, 12, 11, 4, \mathbf{3}.$$

Joshua wonders how many times he should expect to roll the 12-sided die so that he can remove all but 12 of the numbers from the entire sequence of rolls and (without changing the order of the sequence), be left with the sequence

$$1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12.$$

What is the expected value of the number of times Joshua must roll the die before he has such a sequence? (Assume Joshua starts from the beginning - do *not* assume he starts by rolling the specific sequence of 31 rolls above.)

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27 Hannah Kubik leads a local volunteer group of thirteen adults that takes turns holding classes for patients at the Children's Hospital. At the end of August, Hannah took a tour of the hospital and talked with some members of the staff. Dr. Yang told Hannah that it looked like there would be more girls than boys in the hospital during September. The next day Hannah brought the volunteers together and it was decided that three women and two men would volunteer to run the September classes at the Children's Hospital. If there are exactly six women in the volunteer group, how many combinations of three women and two men could Hannah choose from the volunteer group to run the classes?

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28 Of the thirteen members of the volunteer group, Hannah selects herself, Tom Morris, Jerry Hsu, Thelma Paterson, and Louise Bueller to teach the September classes. When she is done, she decides that it's not necessary to balance the number of female and male teachers with the proportions of girls and boys at the hospital *every* month, and having half the women work while only 2 of the 7 men work on some months means that some of the women risk getting burned out. After all, nearly all the members of the volunteer group have other jobs.

Hannah comes up with a plan that the committee likes. Beginning in October, the comittee of five volunteer teachers will consist of any five members of the volunteer group, so long as there is at least one woman and at least one man teaching each month. Under this new plan, what is the least number of months that *must* go by (including October when the first set of five teachers is selected, but not September) such that some five-member comittee *must have* taught together twice (all five members are the same during two different months)?

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29 Find the number of ordered triplets (a, b, c) of positive integers such that $abc = 2008$ (the product of a, b, and c is 2008).

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30 Find the number of ordered triplets (a, b, c) of positive integers such that $a < b < c$ and $abc = 2008$.

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31 The n^{th} term of a sequence is $a_n = (-1)^n(4n + 3)$. Compute the sum

$$a_1 + a_2 + a_3 + \cdots + a_{2008}.$$

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32 A right triangle has perimeter 2008, and the area of a circle inscribed in the triangle is $100\pi^3$. Let A be the area of the triangle. Compute $\lfloor A \rfloor$.

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33 One night, over dinner Jerry poses a challenge to his younger children: "Suppose we travel 50 miles per hour while heading to our final vacation destination..."

Hannah teases her husband, "You *would* drive that *slowly*!"

Jerry smirks at Hannah, then starts over, "So that we get a good view of all the beautiful landscape your mother likes to photograph from the passenger's seat, we travel at a constant rate of 50 miles per hour on the way to the beach. However, on the way back we travel at a faster constant rate along the exact same route. If our faster return rate is an integer number of miles per hour, and our average speed for the *whole round trip* is *also* an integer number of miles per hour, what must be our speed during the return trip?"

Michael pipes up, "How about 4950 miles per hour?!"

Wendy smiles, "For the sake of your *other* children, please don't let *Michael* drive."

Jerry adds, "How about we assume that we never *ever* drive more than 100 miles per hour. Michael and Wendy, let Josh and Alexis try this one."

Joshua ignores the problem in favor of the huge pile of mashed potatoes on his plate. But Alexis scribbles some work on her napkin and declares the correct answer. What answer did Alexis find?

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34 While entertaining his younger sister Alexis, Michael drew two different cards from an ordinary deck of playing cards. Let a be the probability that the cards are of different ranks. Compute $\lfloor 1000a \rfloor$.

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35 Let b be the probability that the cards are from different suits. Compute $\lfloor 1000b \rfloor$.

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36 Let c be the probability that the cards are neither from the same suit or the same rank. Compute $\lfloor 1000c \rfloor$.

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37 A triangle has sides of length 48, 55, and 73. Let a and b be relatively prime positive integers such that a/b is the length of the shortest altitude of the triangle. Find the value of $a+b$.

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38 The volume of a certain rectangular solid is 216 cm^3, its total surface area is 288 cm^2, and its three dimensions are in geometric progression. Find the sum of the lengths in cm of all the edges of this solid.

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39 Let $\phi(n)$ denote *Euler's phi function*, the number of integers $1 \leq i \leq n$ that are relatively prime to n. (For example, $\phi(6) = 2$ and $\phi(10) = 4$.) Let

$$S = \sum_{d|2008} \phi(d),$$

in which d ranges through all positive divisors of 2008, including 1 and 2008. Find the remainder when S is divided by 1000.

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40 Find the number of integers n that satisfy *both* of the following conditions:

-$208 < n < 2008$,
-n has the same remainder when divided by 24 or by 30.

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41 Suppose that

$$x_1 + 1 = x_2 + 2 = x_3 + 3 = \cdots = x_{2008} + 2008 = x_1 + x_2 + x_3 + \cdots + x_{2008} + 2009.$$

Find the value of $\lfloor |S| \rfloor$, where $S = \sum_{n=1}^{2008} x_n$.

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42 Joshua's physics teacher, Dr. Lisi, lives next door to the Kubiks and is a long time friend of the family. An unusual fellow, Dr. Lisi spends as much time surfing and raising chickens as he does trying to map out a *Theory of Everything*. Dr. Lisi often poses problems to the Kubik children to challenge them to think a little deeper about math and science. One day while discussing sequences with Joshua, Dr. Lisi writes out the first 2008 terms of an arithmetic progression that begins $-1776, -1765, -1754, \ldots$. Joshua then computes the (positive) difference between the 1980^{th} term in the sequence, and the 1977^{th} term in the sequence. What number does Joshua compute?

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43 Alexis notices Joshua working with Dr. Lisi and decides to join in on the fun. Dr. Lisi challenges her to compute the sum of all 2008 terms in the sequence. Alexis thinks about the problem and remembers a story one of her teahcers at school taught her about how a young Karl Gauss quickly computed the sum

$$1 + 2 + 3 + \cdots + 98 + 99 + 100$$

in elementary school. Using Gauss's method, Alexis correctly finds the sum of the 2008 terms in Dr. Lisi's sequence. What is this sum?

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44 Now Wendy wanders over and joins Dr. Lisi and her younger siblings. Thinking she knows everything there is about how to work with arithmetic series, she nearly turns right around to walk back home when Dr. Lisi poses a more challenging problem. "Suppose I select two distinct terms at random from the 2008 term sequence. What's the probability that their product is positive?" If a and b are relatively prime positive integers such that a/b is the probability that the product of the two terms is positive, find the value of $a + b$.

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45 In order to save money on gas and use up less fuel, Hannah has a special battery installed in the family van. Before the installation, the van averaged 18 miles per gallon of gas. After the conversion, the van got 24 miles per gallong of gas.

Michael notes, "The amount of money we will save on gas over any time period is equal to the amount we would save if we were able to convert the van to go from 24 miles per gallon to m miles per gallon. It is also the same that we would save if we were able to convert the van to go from m miles per gallon to n miles per gallon."

Assuming Michael is correct, compute $m + n$. In this problem, assume that gas mileage is constant over all speeds and terrain and that the van gets used the same amount regardless of its present state of conversion.

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46 Let S be the sum of all x in the interval $[0, 2\pi)$ that satisfy

$$\tan^2 x - 2\tan x \sin x = 0.$$

Compute $\lfloor 10S \rfloor$.

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47 Find $a + b + c$, where a, b, and c are the hundreds, tens, and units digits of the six-digit number $123abc$, which is a multiple of 990.

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48 Jerry's favorite number is 97. He knows all kinds of interesting facts about 97:

-97 is the largest two-digit prime.
-Reversing the order of its digits results in another prime.
-There is only one way in which 97 can be written as a difference of two perfect squares.
-There is only one way in which 97 can be written as a sum of two perfect squares.
-$\frac{1}{97}$ has exactly 96 digits in the [smallest] repeating block of its decimal expansion.
-Jerry blames the sock gnomes for the theft of exactly 97 of his socks.

A repunit is a natural number whose digits are all 1. For instance,

$$\begin{gathered}1,\\11,\\111,\\1111,\\\vdots\end{gathered}$$

are the four smallest repunits. How many digits are there in the smallest repunit that is divisible by 97?

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49 Wendy takes Honors Biology at school, a smallish class with only fourteen students (including Wendy) who sit around a circular table. Wendy's friends Lucy, Starling, and Erin are also in that class. Last Monday none of the fourteen students were absent from class. Before the teacher arrived, Lucy and Starling stretched out a blue piece of yarn between them. Then Wendy and Erin stretched out a red piece of yarn between them at about the same height so that the yarn would intersect if possible. If all possible positions of the students around the table are equally likely, let m/n be the probability that the yarns intersect, where m and n are relatively prime positive integers. Compute $m + n$.

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50 As the Kubiks head out of town for vacation, Jerry takes the first driving shift while Hannah and most of the kids settle down to read books they brought along. Tony does not feel like reading, so Alexis gives him one of her math notebooks and Tony gets to work solving some of the problems, and struggling over others. After a while, Tony comes to a problem he likes from an old AMC 10 exam:

Four distinct circles are drawn in a plane. What is the maximum number of points where at least two of the circles intersect?

Tony realizes that he can draw the four circles such that each pair of circles intersects in two points. After careful doodling, Tony finds the correct answer, and is proud that he can solve a problem from late on an AMC 10 exam.

"Mom, why didn't we all get Tony's brain?" Wendy inquires before turning he head back into her favorite Harry Potter volume (the fifth year).

Joshua leans over to Tony's seat to see his brother's work. Joshua knows that Tony has not yet discovered all the underlying principles behind the problem, so Joshua challenges, "What if there are a dozen circles?"

Tony gets to work on Joshua's problem of finding the maximum number of points of intersections where at least two of the twelve circles in a plane intersect. What is the answer to this problem?

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51 Alexis imagines a 2008×2008 grid of integers arranged sequentially in the following way:

$$\begin{array}{rrrcr} 1, & 2, & 3, & \ldots, & 2008 \\ 2009, & 2010, & 2011, & \ldots, & 4026 \\ 4017, & 4018, & 4019, & \ldots, & 6024 \\ \vdots & & & & \vdots \\ 2008^2-2008+1, & 2008^2-2008+2, & 2008^2-2008+3, & \ldots, & 2008^2 \end{array}$$

She picks one number from each row so that no two numbers she picks are in the same column. She them proceeds to add them together and finds that S is the sum. Next, she picks 2008 of the numbers that are distinct from the 2008 she picked the first time. Again she picks exactly one number from each row and column, and again the sum of all 2008 numbers is S. Find the remainder when S is divided by 2008.

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52 A triangle has sides of length 48, 55, and 73. A square is inscribed in the triangle such that one side of the square lies on the longest side of the triangle, and the two vertices not on that side of the square touch the other two sides of the triangle. If c and d are relatively prime positive integers such that c/d is the length of a side of the square, find the value of $c+d$.

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53 Find the sum of the 2007 roots of

$$(x-1)^{2007}+2(x-2)^{2006}+3(x-3)^{2005}+\cdots+2006(x-2006)^2+2007(x-2007).$$

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54 One of Michael's responsibilities in organizing the family vacation is to call around and find room rates for hotels along the root the Kubik family plans to drive. While calling hotels near the Grand Canyon, a phone number catches Michael's eye. Michael notices that the first four digits of $987-1234$ descend $(9-8-7-1)$ and that the last four ascend in order $(1-2-3-4)$. This fact along with the fact that the digits are split into consecutive groups makes that number easier to remember.

Looking back at the list of numbers that Michael called already, he notices that several of the phone numbers have the same property: their first four digits are in descending order while the last four are in ascending order. Suddenly, Michael realizes that he can remember all those numbers without looking back at his list of hotel phone numbers. "Wow," he thinks, "that's good marketing strategy."

Michael then wonders to himself how many businesses in a single area code could have such phone numbers. How many 7-digit telephone numbers are there such that all seven digits are distinct, the first four digits are in descending order, and the last four digits are in ascending order?

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55 Let $\triangle XOY$ be a right-angled triangle with $\angle XOY = 90^\circ$. Let M and N be the midpoints of legs OX and OY, respectively. Find the length XY given that $XN = 22$ and $YM = 31$.

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56 During the van ride from the Grand Canyon to the beach, Michael asks his dad about the costs of renewable energy resources. ”How much more does it really cost for a family like ours to switch entirely to renewable energy?”

Jerry explains, ”Part of that depends on where the family lives. In the Western states, solar energy pays off more than it does where we live in the Southeast. But as technology gets better, costs of producing more photovoltaic power go down, so in just a few years more people will have reasonably inexpensive options for switching to clearner power sources. Even now most families could switch to biomass for between $\$200$ and $\$1000$ per year. The energy comes from sawdust, switchgrass, and even landfill gas. We pay that premium ourselves, but some families operate on a tighter budget, or don't understand the alternatives yet.”

”Ew, landfill gas!” Alexis complains mockingly.

Wanting to save her own energy, Alexis decides to take a nap. She falls asleep and dreams of walking around a $2-$D coordinate grid, looking for a wormhole that she believes will transport her to the beach (bypassing the time spent in the family van). In her dream, Alexis finds herself holding a device that she recognizes as a *tricorder* from one of the old *Star Trek* t.v. series. The tricorder has a button labeled ”wormhole” and when Alexis presses the button, a computerized voice from the tricorder announces, ”You are at the origin. Distance to the wormhole is 2400 units. Your wormhole distance allotment is *two*.”'

Unsure as to how to reach, Alexis begins walking forward. As she walks, the tricorder displays at all times her distance from her starting point at the origin. When Alexis is 2400 units from the origin, she again presses the ”wormhole” buttom. The same computerized voice as before begins, ”Distance to the origin is 2400 units. Distance to the wormhole is 3840 units. Your wormhole distance allotment is *two*.”

Alexis begins to feel disoriented. She wonders what is means that her *wormhole distance allotment is two*, and why that number didn't change as she pushed the button. She puts her hat down to mark her position, then wanders aroud a bit. The tricorder shows her two readings as she walks. The first she recognizes as her distance to the origin. The second reading clearly indicates her distance from the point where her hat lies - where she last pressed the button that gave her distance to the wormhole.

Alexis picks up her hat and begins walking around. Eventually Alexis finds herself at a spot 2400 units from the origin and 3840 units from where she last pressed the button. Feeling hopeful, Alexis presses the tricorder's wormhole button again. Nothing happens. She presses it again, and again nothing happens. ”Oh,” she thinks, ”my wormhole allotment was *two*, and I used it up already!”

Despair fills poor Alexis who isn't sure what a wormhole looks like or how she's supposed to find it. Then she takes matters into her own hands. Alexis sits down and scribbles some notes and realizes where the wormhole must be. Alexis gets up and runs straight from her ”third position” to the wormhole. As she gets closer, she sees the wormhole, which looks oddly like a huge scoop of icecream. Alexis runs into the wormhole, then wakes up.

How many units did Alexis run from her third position to the wormhole?

---

57 Let a and b be the two possible values of $\tan\theta$ given that

$$\sin\theta + \cos\theta = \frac{193}{137}.$$

If $a+b=m/n$, where m and n are relatively prime positive integers, compute $m+n$.

---

58 Finished with rereading Isaac Asimov's *Foundation* series, Joshua asks his father, ”Do you think somebody will build small devices that run on nuclear energy while I'm alive?”

”Honestly, Josh, I don't know. There are a lot of very different engineering problems involved in designing such devices. But technology moves forward at an amazing pace, so I won't tell you we can't get there in time for you to see it. I *did* go to a graduate school with a lady who now works on *portable* nuclear reactors. They're not small exactly, but they aren't nearly as large as most reactors. That might be the first step toward a nuclear-powered pocket-sized video game.

Hannah adds, ”There are already companies designing batteries that are nuclear in the sense that they release energy from uranium hydride through controlled exoenergetic processes. This process is not the same as the nuclear fission going on in today's reactors, but we can certainly call it *nuclear energy*.”

”Cool!” Joshua's interest is piqued.

Hannah continues, ”Suppose that right now in the year 2008 we can make one of these nuclear batteries in a battery shape that is 2 meters *across*. Let's say you need that size to be reduced to 2 centimeters *across*, in the same proportions, in order to use it to run your little video game machine. If every year we reduce the necessary volume of such a battery by $1/3$, in what year will the batteries first get small enough?”

Joshua asks, ”The battery shapes never change? Each year the new batteries are similar in shape - in all dimensions - to the bateries from previous years?”

”That's correct,” confirms Joshua's mother. ”Also, the base 10 logarithm of 5 is about 0.69897 and the base 10 logarithm of 3 is around 0.47712.” This makes Joshua blink. He's not sure he knows how to use logarithms, but he does think he can compute the answer. He correctly notes that after 13 years, the batteries will already be barely more than a sixth of their original width.

Assuming Hannah's prediction of volume reduction is correct and effects are compounded continuously, compute the first year that the nuclear batteries get small enough for pocket video game machines. Assume also that the year 2008 is $7/10$ complete.

59 Let a and b be relatively prime positive integers such that

$$\frac{a}{b}=\frac{1}{2^1}+\frac{2}{3^2}+\frac{3}{2^3}+\frac{4}{3^4}+\frac{5}{2^5}+\frac{6}{3^6}+\cdots,$$

where the numerators always increase by 1, and the denominators alternate between powers of 2 and 3, with exponents also increasing by 1 for each subsequent term. Compute $a+b$.

---

60 Consider the Harmonic Table

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & \frac{1}{2} & & \frac{1}{2} & & \\ & \frac{1}{3} & & \frac{1}{6} & & \frac{1}{3} & \\ \frac{1}{4} & & \frac{1}{12} & & \frac{1}{12} & & \frac{1}{4} \\ & & & \vdots & & & \end{array}$$

where $a_{n,1}=1/n$ and

$$a_{n,k+1}=a_{n-1,k}-a_{n,k}.$$

Find the remainder when the sum of the reciprocals of the 2007 terms on the 2007^{th} row gets divided by 2008.

---

61 Find the units digit in the decimal expansion of

$$\left(2008+\sqrt{4032000}\right)^{2000}+\left(2008+\sqrt{4032000}\right)^{2001}+\left(2008+\sqrt{4032000}\right)^{2002}+$$
$$\cdots+\left(2008+\sqrt{4032000}\right)^{2007}+\left(2008+\sqrt{4032000}\right)^{2008}.$$

---

62 Find the number of values of x such that the number of square units in the area of the isosceles triangle with sides x, 65, and 65 is a positive integer.

---

63 Looking for a little time alone, Michael takes a jog at along the beach. The crashing of waves reminds him of the hydroelectric plant his father helped maintain before the family moved to Jupiter Falls. Michael was in elementary school at the time. He thinks about whether he wants to study engineering in college, like both his parents did, or pursue an education in business. His aunt Jessica studied business and appraises budding technology companies for a venture capital firm. Other possibilities also tug a little at Michael for different reasons.

Michael stops and watches a group of girls who seem to be around Tony's age play a game around an ellipse drawn in the sand. There are two softball bats stuck in the sand. Michael recognizes these as the foci of the ellipse. The bats are 24 feet apart. Two children stand on opposite ends of the ellipse where the ellipse intersects the line on which the bats lie. These two children are 40 feet apart. Five other children stand on different points of the ellipse. One of them blows a whistle and all seven children run screaming toward one bat or the other. Each child runs as fast as she can, touching one bat, then the next, and finally returning to the spot on which she started. When the first girl gets back to her place, she declares, ”I win this time! I win!” Another of the girls pats her on the back, and the winning girl speaks again. ”This time I found the place where I'd have to run the shortest distance.”

Michael thinks for a moment, draws some notes in the sand, then computes the shortest possible distance one of the girls could run from her starting point on the ellipse, to one of the bats, to the other bat, then back to her starting point. He smiles for a moment, then keeps jogging. If Michael's work is correct, what distance did he compute as the shortest possible distance one of the girls could run during the game?

---

64 Alexis and Joshua are walking along the beach when they decide to draw symbols in the sand. Alex draws only stars and only draws them in pairs while Joshua draws only squares in trios. ”Let's see how many rows of 15 adjacent symbols we can make this way,” suggests Josh. Alexis is game for the task and the two get busy drawing. Some of their rows look like

∗ ∗ ∗ ∗ ∗ ∗ ∗ ∗ ∗ ∗ ∗ ∗ ■ ■ ■

■ ■ ■ ∗ ∗ ∗ ∗ ∗ ∗ ∗ ∗ ∗ ∗ ∗ ∗

■ ■ ■ ■ ■ ■ ∗ ∗ ∗ ∗ ∗ ∗ ■ ■ ■

■ ■ ■ ■ ■ ■ ■ ■ ■ ■ ■ ■ ■ ■ ■

∗ ∗ ∗ ∗ ∗ ∗ ■ ■ ■ ∗ ∗ ∗ ∗ ∗ ∗

The twins decide to count each of the first two rows above as distinct rows, even though one is the mirror image of the other. But the last of the rows above is its own mirror image, so they count it only once. Around an hour later, the twins realized that they had drawn every possible row exactly once using their rules of stars in pairs and squares in trips. How many rows did they draw in the sand?

---

65 Just as the twins finish their masterpiece of symbol art, Wendy comes along. Wendy is impressed by the explanation Alexis and Joshua give her as to how they knew they drew every row exactly once. Wendy puts them both to the test. ”Suppose the two of you draw symbols as you have before, stars in pairs and boxes in threes.” Wendy continues, ”Now, suppose that

I draw circles with X's in the middle." Wendy shows them examples of such rows:

$$\ast \quad \ast \quad \ast \quad \ast \quad \otimes \quad \ast \quad \ast \quad \otimes \quad \ast \quad \ast \quad \ast \quad \ast \quad \blacksquare \quad \blacksquare \quad \blacksquare$$

$$\blacksquare \quad \blacksquare \quad \blacksquare \quad \ast \quad \ast \quad \ast \quad \ast \quad \otimes \quad \ast \quad \ast \quad \otimes \quad \ast \quad \ast \quad \ast \quad \ast$$

$$\otimes \quad \blacksquare \quad \blacksquare \quad \blacksquare \quad \otimes \quad \otimes \quad \ast \quad \ast \quad \otimes \quad \ast \quad \ast \quad \otimes \quad \blacksquare \quad \blacksquare \quad \blacksquare$$

"Again we count both the first two rows, which are mirror images of one another, but we only count a row that is its own mirror image. *Now* how man rows of 15 symbols are possible?"

Though it takes the twins some time, they eventually come up with an answer they agree on. Wendy confirms that they are correct. How many rows did the twins find are possible using all three symbols as described?

---

66 Michael draws $\triangle ABC$ in the sand such that $\angle ACB = 90^\circ$ and $\angle CBA = 15^\circ$. He then picks a point at random from within the triangle and labels it point M. Next, he draws a segment from A to BC that passes through M, hitting BC at a point he labels D. Just then, a wave washes over his work, so Michael redraws the exact same diagram farther from the water, labeling all the points the same way as before. If hypotenuse AB is 4 feet in length, let p be the probability that the number of feet in the length of AD is less than $2\sqrt{3}-2$. Compute $\lfloor 1000p \rfloor$.

---

67 At lunch, the seven members of the Kubik family sit down to eat lunch together at a round table. In how many distinct ways can the family sit at the table if Alexis refuses to sit next to Joshua? (Two arrangements are not considered distinct if one is a rotation of the other.)

---

68 Let u_n be the n^{th} term of the sequence

$$1, \quad 2, \quad 5, \quad 6, \quad 9, \quad 12, \quad 13, \quad 16, \quad 19, \quad 22, \quad 23, \ldots,$$

where the first term is the smallest positive integer that is 1 more than a multiple of 3, the next two terms are the next two smallest positive integers that are each two more than a multiple of 3, the next three terms are the next three smallest positive integers that are each three more than a multiple of 3, the next four terms are the next four smallest positive integers that are each four more than a multiple of 3, and so on:

$$\underbrace{1}_{\text{1 term}}, \quad \underbrace{2, \quad 5,}_{\text{2 terms}} \quad \underbrace{6, \quad 9, \quad 12,}_{\text{3 terms}} \quad \underbrace{13, \quad 16, \quad 19, \quad 22,}_{\text{4 terms}} \quad \underbrace{23, \ldots,}_{\text{5 terms}} \quad \ldots.$$

Determine u_{2008}.

---

69 In the sequence in the previous problem, how many of $u_1, u_2, u_3, \ldots, u_{2008}$ are pentagonal numbers?

---

70 After swimming around the ocean with some snorkling gear, Joshua walks back to the beach where Alexis works on a mural in the sand beside where they drew out symbol lists. Joshua walks directly over the mural without paying any attention.

"You're a square, Josh."

"No, *you're* a square," retorts Joshua. "In fact, you're a *cube*, which is 50% freakier than a square by dimension. And before you tell me I'm a hypercube, I'll remind you that mom and dad confirmed that they could not have given birth to a four dimension being."

"Okay, you're a cubist caricature of male immaturity," asserts Alexis.

Knowing nothing about cubism, Joshua decides to ignore Alexis and walk to where he stashed his belongings by a beach umbrella. He starts thinking about cubes and computes some sums of cubes, and some cubes of sums:

$$\begin{aligned}
1^3+1^3+1^3&=3,\\
1^3+1^3+2^3&=10,\\
1^3+2^3+2^3&=17,\\
2^3+2^3+2^3&=24,\\
1^3+1^3+3^3&=29,\\
1^3+2^3+3^3&=36,\\
(1+1+1)^3&=27,\\
(1+1+2)^3&=64,\\
(1+2+2)^3&=125,\\
(2+2+2)^3&=216,\\
(1+1+3)^3&=125,\\
(1+2+3)^3&=216.
\end{aligned}$$

Josh recognizes that the cubes of the sums are always larger than the sum of cubes of positive integers. For instance,

$$\begin{aligned}
(1+2+4)^3&=1^3+2^3+4^3+3(1^2\cdot 2+1^2\cdot 4+2^2\cdot 1+2^2\cdot 4+4^2\cdot 1+4^2\cdot 2)+6(1\cdot 2\cdot 4)\\
&>1^3+2^3+4^3.
\end{aligned}$$

Josh begins to wonder if there is a smallest value of n such that

$$(a+b+c)^3\leq n(a^3+b^3+c^3)$$

for all natural numbers a, b, and c. Joshua thinks he has an answer, but doesn't know how to prove it, so he takes it to Michael who confirms Joshua's answer with a proof. What is the correct value of n that Joshua found?

---

71 One day Joshua and Alexis find their sister Wendy's copy of the 2007 iTest. They decide to see if they can work any of the problems and are proud to find that indeed they are able to work some of them, but their middle school math team experience is still not enough to help with the harder problems.

Alexis comes across a problem she really likes, partly because she has never worked one like it before:

What is the smallest positive integer k such that the number $\binom{2k}{k}$ ends in two zeroes?

Joshua is the kind of mathematical explorer who likes to alter problems, make them harder, or generalize them. So, he proposes the following problem to his sister Alexis:

What is the smallest positive integer k such that the number $\binom{2k}{k}$ ends in two zeroes when expressed in

Alexis solves the problem correctly. What is her answer (expressed in base 10)?

---

72 On the last afternoon of the Kubik family vacation, Michael puts down a copy of *Mathematical Olympiad Cha* and goes out to play tennis. Wendy notices the book and decides to see if there are a few problems in it that she can solve. She decides to focus her energy on one problem in particular:

> Given 69 distinct positive integers not exceeding 100, prove that one can choose four of them a, b, c, d such that $a < b < c$ and $a + b + c = d$. Is this statement true for 68 numbers?

After some time working on the problem, Wendy finally feels like she has a grip on the solution. When Michael returns, she explains her solutions to him. "Well done!" he tells her. "Now, see if you can solve this generalization. Consider the set

$$S = \{1, 2, 3, \ldots, 2007, 2008\}.$$

Find the smallest value of t such that given any subset T of S where $|T| = t$, then there are necessarily distinct $a, b, c, d \in T$ for which $a + b + c = d$." Find the answer to Michael's generalization.

---

73 As the Kubiks head homeward, away from the beach in the family van, Jerry decides to take a different route away from the beach than the one they took to get there. The route involves lots of twists and turns, prompting Hannah to wonder aloud if Jerry's "shortcut" will save any time at all.

Michael offers up a problem as an analogy to his father's meandering: "Suppose dad drives around, making right-angled turns after *every* mile. What is the farthest he could get us from our starting point after driving us 500 miles assuming that he makes exactly 300 right turns?"

"Sounds almost like an energy efficiency problem," notes Hannah only half jokingly. Hannah is always encouraging her children to think along these lines.

Let d be the answer to Michael's problem. Compute $\lfloor d \rfloor$.

---

74 Points C and D lie on opposite sides of line $\overline{AB}$. Let M and N be the centroids of $\triangle ABC$ and $\triangle ABD$ respectively. If $AB = 841$, $BC = 840$, $AC = 41$, $AD = 609$, and $BD = 580$, find the sum of the numerator and denominator of the value of MN when expressed as a fraction in lowest terms.

---

75 Let

$$S = \sqrt{1 + \frac{1}{1^2} + \frac{1}{2^2}} + \sqrt{1 + \frac{1}{2^2} + \frac{1}{3^2}} + \cdots + \sqrt{1 + \frac{1}{2007^2} + \frac{1}{2008^2}}.$$

Compute $\lfloor S^2 \rfloor$.

---

76 During the car ride home, Michael looks back at his recent math exams. A problem on Michael's calculus mid-term gets him starting thinking about a particular quadratic,

$$x^2 - sx + p,$$

with roots r_1 and r_2. He notices that

$$r_1 + r_2 = r_1^2 + r_2^2 = r_1^3 + r_2^3 = \cdots = r_1^{2007} + r_2^{2007}.$$

He wonders how often this is the case, and begins exploring other quantities associated with the roots of such a quadratic. He sets out to compute the greatest possible value of

$$\frac{1}{r_1^{2008}} + \frac{1}{r_2^{2008}}.$$

Help Michael by computing this maximum.

---

77 With about six hours left on the van ride home from vacation, Wendy looks for something to do. She starts working on a project for the math team.

There are sixteen students, including Wendy, who are about to be sophomores on the math team. Elected as a math team officer, one of Wendy's jobs is to schedule groups of the sophomores to tutor geometry students after school on Tuesdays. The way things have been done in the past, the same number of sophomores tutor every week, but the same group of students never works together. Wendy notices that there are even numbers of groups she could select whether she chooses 4 or 5 students at a time to tutor geometry each week:

$$\binom{16}{4} = 1820,$$
$$\binom{16}{5} = 4368.$$

Playing around a bit more, Wendy realizes that unless she chooses all or none of the students on the math team to tutor each week that the number of possible combinations of the sophomore math teamers is always even. This gives her an idea for a problem for the 2008 Jupiter Falls High School Math Meet team test:

How many of the 2009 numbers on Row 2008 of Pascal's Triangle are even?

Wendy works the solution out correctly. What is her answer?

---

78 Feeling excited over her successful explorations into Pascal's Triangle, Wendy formulates a second problem to use during a future Jupiter Falls High School Math Meet:

How many of the first 2010 rows of Pascal's Triangle (Rows 0 through 2009) have exactly 256 odd entrie

What is the solution to Wendy's second problem?

---

79 Done with her new problems, Wendy takes a break from math. Still without any fresh reading material, she feels a bit antsy. She starts to feel annoyed that Michael's loose papers clutter the family van. Several of them are ripped, and bits of paper litter the floor. Tired of trying to get Michael to clean up after himself, Wendy spends a couple of minutes putting Michael's loose papers in the trash. "That seems fair to me," confirms Hannah encouragingly.

While collecting Michael's scraps, Wendy comes across a corner of a piece of paper with part of a math problem written on it. There is a monic polynomial of degree n, with real coefficients. The first two terms after x^n are $a_{n-1}x^{n-1}$ and $a_{n-2}x^{n-2}$, but the rest of the polynomial is cut off where Michael's page is ripped. Wendy barely makes out a little of Michael's scribbling, showing that $a_{n-1} = -a_{n-2}$. Wendy deciphers the goal of the problem, which is to find the sum of the squares of the roots of the polynomial. Wendy knows neither the value of n, nor the value of a_{n-1}, but still she finds a [greatest] lower bound for the answer to the problem. Find the absolute value of that lower bound.

---

80 Let

$$p(x) = x^{2008} + x^{2007} + x^{2006} + \cdots + x + 1,$$

and let $r(x)$ be the polynomial remainder when $p(x)$ is divided by $x^4+x^3+2x^2+x+1$. Find the remainder when $|r(2008)|$ is divided by 1000.

---

81 Compute the number of 7-digit positive integers that start *or* end (or both) with a digit that is a (nonzero) composite number.

---

82 Tony's favorite "sport" is a spectator event known as the *Super Mega Ultra Galactic Thumbwrestling Champic* (SMUG TWC). During the 2008 SMUG TWC, 2008 professional thumb-wrestlers who have dedicated their lives to earning lithe, powerful thumbs, compute to earn the highest title of *Thumbzilla*. The SMUG TWC is designed so that, in the end, any set of three participants can share a banana split while telling FOX™ television reporters about a bout between some pair of the three contestants. Given that there are exactly two contestants in each bout, let m be the minimum bumber of bouts necessary to complete the SMUG TWC (so that the contestants can enjoy their banana splits and chat with reporters). Compute m.

---

83 Find the greatest natural number n such that $n\leq 2008$ and

$$(1^2+2^2+3^2+\cdots+n^2)\left[(n+1)^2+(n+2)^2+(n+3)^2+\cdots+(2n)^2\right]$$

is a perfect square.

---

84 Let S be the sum of all integers b for which the polynomial $x^2+bx+2008b$ can be factored over the integers. Compute $|S|$.

---

85 Let (a,b,c,d) be a solution to the system

$$\begin{aligned} a+b&=15,\\ ab+c+d&=78,\\ ad+bc&=160,\\ cd&=96. \end{aligned}$$

Find the greatest possible value of $a^2+b^2+c^2+d^2$.

---

86 Let a, b, c, and d be positive real numbers such that

$$\begin{array}{rcl} a^2+b^2 &=& c^2+d^2 = 2008,\\ ac &=& bd \quad = 1000. \end{array}$$

If $S=a+b+c+d$, compute the value of $\lfloor S\rfloor$.

---

87 Find the number of 12-digit "words" that can be formed from the alphabet $\{0,1,2,3,4,5,6\}$ if neighboring digits must differ by exactly 2.

---

88 A six dimensional "cube" (a 6-cube) has 64 vertices at the points $(\pm 3, \pm 3, \pm 3, \pm 3, \pm 3, \pm 3)$. This 6-cube has 192 1-D edges and 240 2-D edges. This 6-cube gets cut into $6^6 = 46656$ smaller congruent "unit" 6-cubes that are kept together in the tightly packaged form of the original 6-cube so that the 46656 smaller 6-cubes share 2-D square faces with neighbors (*one* 2-D square face shared by *several* unit 6-cube neighbors). How many 2-D squares are faces of one or more of the unit 6-cubes?

---

89 Two perpendicular planes intersect a sphere in two circles. These circles intersect in two points, A and B, such that $AB = 42$. If the radii of the two circles are 54 and 66, find R^2, where R is the radius of the sphere.

---

90 For a, b, c positive reals, let

$$N = \frac{a^2+b^2}{c^2+ab} + \frac{b^2+c^2}{a^2+bc} + \frac{c^2+a^2}{b^2+ca}.$$

Find the minimum value of $\lfloor 2008N \rfloor$.

---

91 Find the sum of all positive integers n such that

$$x^3 + y^3 + z^3 = nx^2y^2z^2$$

is satisfied by at least one ordered triplet of positive integers (x, y, z).

---

92 Find [the decimal form of] the largest prime divisor of 100111011_6.

---

93 For how many positive integers n, $1 \leq n \leq 2008$, can the set

$$\{1, 2, 3, \ldots, 4n\}$$

be divided into n disjoint 4-element subsets such that every one of the n subsets contains the element which is the arithmetic mean of all the elements in that subset?

---

94 Find the largest prime number less than 2008 that is a divisor of some integer in the infinite sequence

$$\left\lfloor \frac{2008}{1} \right\rfloor, \quad \left\lfloor \frac{2008^2}{2} \right\rfloor, \quad \left\lfloor \frac{2008^3}{3} \right\rfloor, \quad \left\lfloor \frac{2008^4}{4} \right\rfloor, \quad \ldots.$$

---

95 Bored on their trip home, Joshua and Alexis decide to keep a tally of license plates they see in the other lanes: Joshua watches cars going the other way, and Alexis watches cars in the next lane.

After a few minutes, Wendy counts up the tallies and declares, "Joshua has counted 2008 license plates, and there are 17 license plate designs he's seen exactly 17 times, but of Alexis's

2009 license plates, there's none she's seen exactly 18 times. Clearly, 17 is the specialist number."

Michael, suspicious, pulls out the *Almanac of American License Plates* and notes, "According to confirmed demographic statistics, you'd only expect those numbers to be 5.4 and 4.9, respectively. But the 17^{th} state is weird: Joshua saw exactly 17 of its license plates, which isn't what we'd expect."

Alexis asks, "How many Ohioan license plates did we expect to see?" and reaches for the *Almanac* to find out, but Michael snatches it away and says, "I'm not telling."

Alexis, disappointed, says, "I suppose that 17 is my best guess," feeling that the answer must be pretty close to 17.

Wendy smiles. "You can do better than that, actually. Given what Michael said and that we saw 17 Ohioan license plates, we'd actually expect there to have been $\frac{a}{b}$ less than 17."

Help Alexis. If $\frac{a}{b}$ is in lowest terms, find the product ab.

---

96 Triangle ABC has $\angle A = 90^\circ$, $\angle B = 60^\circ$, and $AB = 8$, and a point P is chosen inside the triangle. The interior angle bisectors ℓ_A, ℓ_B, and ℓ_C of respective angles PAB, PBC, and PCA intersect pairwise at $X = \ell_A \cap \ell_B$, $Y = \ell_B \cap \ell_C$, and $Z = \ell_C \cap \ell_A$. If triangles ABC and XYZ are directly similar, then the area of $\triangle XYZ$ may be written in the form $\frac{p\sqrt{q}-r\sqrt{s}}{t}$, where p, q, r, s, t are positive integers, q and s are not divisible by the square of any prime, and $\gcd(t, r, p) = 1$. Compute $p + q + r + s + t$.

---

97 During the first week of the 2008-2009 school year at Jupiter Falls High School, the school holds a fire drill. The 2008 students in attendance all leave the school and head for the football field. Wendy and several of her friends sit down in a circle on the ground and begin to chat.

Wendy and her friend Lilly sit side-by-side, and after a little while decide to swap spots in order to make it easier to talk with different friends. This leads Lilly's boyfriend Nori to offer up a problem, "Suppose we all stood up and took the space that one of our neighboors had been sitting in. In how many ways could we do that?"

"I think just four, " offers Wendy, oblivious that Nori is subtly voicing a complaint over Lilly's absence at his side. "We all either move one spot clockwise, or one spot counterclockwise. Unless we can sit on each other."

Nori replies, "Oh, right. That's not really what I meant. What I meant was that we can also stay in our own spot, like Beth, Regan, Tom, Burt, and I just did. So, in how many ways can that happen? Assume no two people wind up in the same spot."

Wendy pulls out a calculator and writes a program that cycles through all the possibilities. After a couple of minutes she announces, "There are 31. *That's* a weird number."

"Can you solve it generally?" asks Lilly.

"Honestly, I'm not sure. I'd need to work on it a bit to know if I could," admits Wendy.

Nori adds more complexity to the problem, "How about this: Let k be the number of students in a circle. Then let m be the number of ways we can rearrange ourselves so that each of us is in the same spot or within one spot of where we started, and no two people are ever in the same spot. If m leaves a remainder of 1 when divided by 5, how many possible values are there of k, where k is at least 3 and at most 2008?"

Find the answer to Nori's problem.

---

98 Convex quadrilateral $ABCD$ has side-lengths $AB = 7$, $BC = 9$, $CD = 15$, and there exists a circle, lying inside the quadrilateral and having center I, that is tangent to all four sides of the quadrilateral. Points M and N are the midpoints of AC and BD respectively. It can be proven that point I always lies on segment MN. Supposing further that I is the midpoint of MN, the area of quadrilateral $ABCD$ may be expressed as $p\sqrt{q}$, where p and q are positive integers and q is not divisible by the square of any prime. Compute $p \cdot q$.

---

99 Given a convex, n-sided polygon P, form a $2n$-sided polygon clip(P) by cutting off each corner of P at the edges' trisection points. In other words, clip(P) is the polygon whose vertices are the $2n$ edge trisection points of P, connected in order around the boundary of P. Let P_1 be an isosceles trapezoid with side lengths $13, 13, 13$, and 3, and for each $i \geq 2$, let $P_i =$ clip(P_{i-1}). This iterative clipping process approaches a limiting shape $P_\infty = \lim_{i\to\infty} P_i$. If the difference of the areas of P_{10} and P_∞ is written as a fraction $\frac{x}{y}$ in lowest terms, calculate the number of positive integer factors of $x \cdot y$.

---

100 Let α be a root of $x^6 - x - 1$, and call two polynomials p and q with integer coefficients *equivalent* if $p(\alpha) \equiv q(\alpha) \pmod 3$. It is known that every such polynomial is equivalent to exactly one of $0, 1, x, x^2, \ldots, x^{727}$. Find the largest integer $n < 728$ for which there exists a polynomial p such that $p^3 - p - x^n$ is equivalent to 0.
