Romania Team Selection Test 1987

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by Valentin Vornicu, schulmannerism, Lagrangia, Arne, grobber, dblues

Day 1 June 8th

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1 Let a,b,c be distinct real numbers such that $a+b+c>0$. Let M be the set of 3×3 matrices with the property that each line and each column contain all given numbers a,b,c. Find $\{\max\{\det A \mid A \in M\}$ and the number of matrices which realise the maximum value.

Mircea Becheanu

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2 Find all positive integers A which can be represented in the form:

$$A=\left(m-\frac{1}{n}\right)\left(n-\frac{1}{p}\right)\left(p-\frac{1}{m}\right)$$

where $m\geq n\geq p\geq 1$ are integer numbers.

Ioan Bogdan

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3 Let A be the set $A=\{1,2,\ldots,n\}$. Determine the maximum number of elements of a subset $B\subset A$ such that for all elements x,y from B, $x+y$ cannot be divisible by $x-y$.

Mircea Lascu, Dorel Mihet

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4 Let $P(X)=a_nX^n+a_{n-1}X^{n-1}+\ldots+a_1X+a_0$ be a real polynomial of degree n. Suppose n is an even number and:

a) $a_0>0$, $a_n>0$;

b) $a_1^2+a_2^2+\ldots+a_{n-1}^2\leq\frac{4\min(a_0^2,a_n^2)}{n-1}$.

Prove that $P(x)\geq 0$ for all real values x.

Laurentiu Panaitopol

Day 2 June 9th

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5 Let A be the set $\{1,2,\ldots,n\}$, $n\geq 2$. Find the least number n for which there exist permutations

α, β, γ, δ of the set A with the property:

$$\sum_{i=1}^{n} \alpha(i)\beta(i) = \frac{19}{10} \sum_{i=1}^{n} \gamma(i)\delta(i).$$

Marcel Chirita

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6 The plane is covered with network of regular congruent disjoint hexagons. Prove that there cannot exist a square which has its four vertices in the vertices of the hexagons.

Gabriel Nagy

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7 Determine all positive integers n such that n divides $3^n - 2^n$.

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8 Let $ABCD$ be a square and a be the length of his edges. The segments AE and CF are perpendicular on the square's plane in the same half-space and they have the length $AE = a$, $CF = b$ where $a < b < a\sqrt{3}$. If K denoted the set of the interior points of the square $ABCD$ determine $\min_{M \in K} (\max(EM, FM))$ and $\max_{M \in K} (\min(EM, FM))$.

Octavian Stanasila

Day 3 June 10th

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9 Prove that for all real numbers $\alpha_1, \alpha_2, \dots, \alpha_n$ we have

$$\sum_{i=1}^{n} \sum_{j=1}^{n} ij \cos(\alpha_i - \alpha_j) \geq 0.$$

Octavian Stanasila

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10 Let a, b, c be integer numbers such that $(a+b+c) \mid (a^2+b^2+c^2)$. Show that there exist infinitely many positive integers n such that $(a+b+c) \mid (a^n + b^n + c^n)$.

Laurentiu Panaitopol

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11 Let $P(X, Y) = X^2 + 2aXY + Y^2$ be a real polynomial where $|a| \geq 1$. For a given positive integer n, $n \geq 2$ consider the system of equations:

$$P(x_1, x_2) = P(x_2, x_3) = \ldots = P(x_{n-1}, x_n) = P(x_n, x_1) = 0.$$

We call two solutions $(x_1, x_2, \dots, x_n)$ and $(y_1, y_2, \dots, y_n)$ of the system to be equivalent if there exists a real number $\lambda \neq 0$, $x_1 = \lambda y_1$, ..., $x_n = \lambda y_n$. How many nonequivalent solutions does the system have?

Mircea Becheanu
