Romania Team Selection Test 1988

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Day 1 April 15th

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1 Consider a sphere and a plane π. For a variable point $M \in \pi$, exterior to the sphere, one considers the circular cone with vertex in M and tangent to the sphere. Find the locus of the centers of all circles which appear as tangent points between the sphere and the cone.

Octavian Stanasila

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2 Let $OABC$ be a trihedral angle such that

$$\angle BOC = \alpha, \quad \angle COA = \beta, \quad \angle AOB = \gamma, \quad \alpha + \beta + \gamma = \pi.$$

For any interior point P of the trihedral angle let P_1, P_2 and P_3 be the projections of P on the three faces. Prove that $OP \geq PP_1 + PP_2 + PP_3$.

Constantin Cocea

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3 Consider all regular convex and star polygons inscribed in a given circle and having n *sides*. We call two such polygons to be equivalent if it is possible to obtain one from the other using a rotation about the center of the circle. How many classes of such polygons exist?

Mircea Becheanu

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4 Prove that for all positive integers $0 < a_1 < a_2 < \cdots < a_n$ the following inequality holds:

$$(a_1 + a_2 + \cdots + a_n)^2 \leq a_1^3 + a_2^3 + \cdots + a_n^3.$$

Viorel Vajaitu

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5 The cells of a 11×11 chess-board are colored in 3 colors. Prove that there exists on the board a $m \times n$ rectangle such that the four cells interior to the rectangle and containing the four vertices of the rectangle have the same color.

Ioan Tomescu

Day 2 June 10th

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6 Find all vectors of n real numbers $(x_1, x_2, \dots, x_n)$ such that

$$\begin{cases} x_1 &= \dfrac{1}{x_2} + \dfrac{1}{x_3} + \cdots + \dfrac{1}{x_n} \\ x_2 &= \dfrac{1}{x_1} + \dfrac{1}{x_3} + \cdots + \dfrac{1}{x_n} \\ & \cdots \\ x_n &= \dfrac{1}{x_1} + \dfrac{1}{x_2} + \cdots + \dfrac{1}{x_{n-1}} \end{cases}$$

Mircea Becheanu

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7 In the plane there are given the lines ℓ_1, ℓ_2, the circle $\mathcal{C}$ with its center on the line ℓ_1 and a second circle $\mathcal{C}_1$ which is tangent to ℓ_1, ℓ_2 and $\mathcal{C}$. Find the locus of the tangent point between $\mathcal{C}$ and $\mathcal{C}_1$ while the center of $\mathcal{C}$ is variable on ℓ_1.

Mircea Becheanu

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8 The positive integer n is given and for all positive integers k, $1 \leq k \leq n$, denote by a_{kn} the number of all ordered sequences $(i_1, i_2, \dots, i_k)$ of positive integers which verify the following two conditions:

a) $1 \leq i_1 < i_2 < \cdots i_k \leq n$;

b) $i_{r+1} - i_r \equiv 1 \pmod 2$, for all $r \in \{1, 2, \dots, k-1\}$.

Compute the number $a(n) = \sum\limits_{k=1}^{n} a_{kn}$.

Ioan Tomescu

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9 Prove that for all positive integers $n \geq 1$ the number $\prod_{k=1}^{n} k^{2k-n-1}$ is also an integer number.

Laurentiu Panaitopol.

Day 3 June 11th

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10 Let $p > 2$ be a prime number. Find the least positive number a which can be represented as

$$a = (X-1)f(X) + (X^{p-1} + X^{p-2} + \cdots + X + 1)g(X),$$

where $f(X)$ and $g(X)$ are integer polynomials.

Mircea Becheanu.

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11 Let x, y, z be real numbers with $x + y + z = 0$. Prove that

$$|\cos x| + |\cos y| + |\cos z| \geq 1.$$

Viorel Vajaitu, Bogdan Enescu

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12 The four vertices of a square are the centers of four circles such that the sum of theirs areas equals the square's area. Take an arbitrary point in the interior of each circle. Prove that the four arbitrary points are the vertices of a convex quadrilateral.

Laurentiu Panaitopol

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13 Let a be a positive integer. The sequence $\{x_n\}_{n\geq 1}$ is defined by $x_1 = 1$, $x_2 = a$ and $x_{n+2} = ax_{n+1} + x_n$ for all $n \geq 1$. Prove that (y, x) is a solution of the equation

$$|y^2 - axy - x^2| = 1$$

if and only if there exists a rank k such that $(y, x) = (x_{k+1}, x_k)$.

Serban Buzeteanu

Day 4 June 12th

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14 Let Δ denote the set of all triangles in a plane. Consider the function $f : \Delta \to (0, \infty)$ defined by $f(ABC) = \min\left(\dfrac{b}{a}, \dfrac{c}{b}\right)$, for any triangle ABC with $BC = a \leq CA = b \leq AB = c$. Find the set of values of f.

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15 Let $[a, b]$ be a given interval of real numbers not containing integers. Prove that there exists $N > 0$ such that $[Na, Nb]$ does not contain integer numbers and the length of the interval $[Na, Nb]$ exceedes $\dfrac{1}{6}$.

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16 The finite sets A_1, A_2, $\ldots$, A_n are given and we denote by $d(n)$ the number of elements which appear exactly in an odd number of sets chosen from A_1, A_2, $\ldots$, A_n. Prove that for any k, $1 \leq k \leq n$ the number

$$d(n) - \sum_{i=1}^{n} |A_i| + 2\sum_{i<j} |A_i \cap A_j| - \cdots + (-1)^k 2^{k-1} \sum_{i_1<i_2<\cdots<i_k} |A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}|$$

is divisible by 2^k.

Ioan Tomescu, Dragos Popescu
