Romania Team Selection Test 1990

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BMO TST

1 Let $f : N \to N$ be a function such that the set $\{k|f(k) < k\}$ is finite.
Prove that the set $\{k|g(f(k)) \leq k\}$ is infinite for all functions $g : N \to N$.

2 Prove that in any triangle ABC the following inequality holds:

$$\frac{a^2}{b+c-a} + \frac{b^2}{a+c-b} + \frac{c^2}{a+b-c} \geq 3\sqrt{3}R.$$

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3 Prove that for any positive integer n, the least common multiple of the numbers $1, 2, \dots, n$ and the least common multiple of the numbers:

$$\binom{n}{1}, \binom{n}{2}, \dots, \binom{n}{n}$$

are equal if and only if $n + 1$ is a prime number.

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4 Let M be a point on the edge CD of a tetrahedron $ABCD$ such that the tetrahedra $ABCM$ and $ABDM$ have the same total areas. We denote by π_{AB} the plane ABM. Planes $\pi_{AC}, ..., \pi_{CD}$ are analogously defined. Prove that the six planes $\pi_{AB}, ..., \pi_{CD}$ are concurrent in a certain point N, and show that N is symmetric to the incenter I with respect to the barycenter G.

IMO TST

Day 1

1 Let a,b,n be positive integers such that $(a, b) = 1$.
Prove that if (x, y) is a solution of the equation $ax + by = a^n + b^n$ then

$$\left[\frac{x}{b}\right] + \left[\frac{y}{a}\right] = \left[\frac{a^{n-1}}{b}\right] + \left[\frac{b^{n-1}}{a}\right]$$

2 Prove the following equality for all positive integers m, n:

$$\sum_{k=0}^{n}\binom{m+k}{k}2^{n-k}+\sum_{k=0}^{m}\binom{n+k}{k}2^{m-k}=2^{m+n+1}$$

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3 Find all polynomials $P(x)$ such that $2P(2x^2-1)=P(x)^2-1$ for all x.

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4 The six faces of a hexahedron are quadrilaterals. Prove that if seven its vertices lie on a sphere, then the eighth vertex also lies on the sphere.

Day 2

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5 Let O be the circumcenter of an acute triangle ABC and R be its circumcenter. Consider the disks having OA, OB, OC as diameters, and let Δ be the set of points in the plane belonging to at least two of the disks. Prove that the area of Δ is greater than $R^2/8$.

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6 Prove that there are infinitely many n's for which there exists a partition of $\{1, 2, ..., 3n\}$ into subsets $\{a_1, ..., a_n\}, \{b_1, ..., b_n\}, \{c_1, ..., c_n\}$ such that $a_i + b_i = c_i$ for all i, and prove that there are infinitely many n's for which there is no such partition.

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7 The sequence $(x_n)_{n\geq 1}$ is defined by: $x_1 = 1$ $x_{n+1} = \frac{x_n}{n} + \frac{n}{x_n}$
Prove that (x_n) increases and $[x_n^2] = n$.

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8 For a set S of n points, let $d_1 > d_2 > ... > d_k > ...$ be the distances between the points.
A function $f_k : S \to N$ is called a *coloring function* if, for any pair M, N of points in S with $MN \leq d_k$, it takes the value $f_k(M) + f_k(N)$ at some point. Prove that for each $m \in N$ there are positive integers n, k and a set S of n points such that every coloring function f_k of S satisfies $|f_k(S)| \leq m$

Day 3

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9 The distance between any two of six given points in the plane is at least 1. Prove that the distance between some two points is at least $\sqrt{\frac{5+\sqrt{5}}{2}}$

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10 Let p, q be positive prime numbers and suppose $q > 5$. Prove that if $q \mid 2^p + 3^p$, then $q > p$.

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11 In a group of n persons,
(i) each person is acquainted to exactly k others,
(ii) any two acquainted persons have exactly l common acquaintances,

(iii) any two non-acquainted persons have exactly m common acquaintances.
Prove that $m(n-k-1)=k(k-l-1)$.
