Romania Team Selection Test 1991

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BMO TST

1 Suppose that a,b are positive integers for which $A=\frac{a+1}{b}+\frac{b}{a}$ is an integer.Prove that $A=3$.

2 Let $A_1A_2A_3A_4$ be a tetrahedron. For any permutation (i,j,k,h) of $1,2,3,4$ denote:
- P_i the orthogonal projection of A_i on $A_jA_kA_h$;
- B_{ij} the midpoint of the edge A_iAj,
- C_{ij} the midpoint of segment P_iP_j
- β_{ij} the plane $B_{ij}P_hP_k$
- δ_{ij} the plane $B_{ij}P_iP_j$
- α_{ij} the plane through C_{ij} orthogonal to A_kA_h
- γ_{ij} the plane through C_{ij} orthogonal to A_iA_j.

Prove that if the points P_i are not in a plane, then the following sets of planes are concurrent: (a) α_{ij}, (b) β_{ij}, (c) γ_{ij}, (d) δ_{ij}.

3 Prove the following identity for every $n\in N$: $\sum_{j+h=n,j\geq h}\frac{(-1)^h2^{j-h}\binom{j}{h}}{j}=\frac{2}{n}$

4 A sequence (a_n) of positive integers satisfies$(a_m,a_n)=a_{(m,n)}$ for all m,n.
Prove that there is a unique sequence (b_n) of positive integers such that $a_n=\prod_{d|n}b_d$

IMO TST

Day 1

1 Let $M=\{A_1,A_2,\ldots,A_5\}$ be a set of five points in the plane such that the area of each triangle $A_iA_jA_k$, is greater than 3. Prove that there exists a triangle with vertices in M and having the area greater than 4.

Laurentiu Panaitopol

2 The sequence (a_n) is defined by $a_1=a_2=1$ and $a_{n+2}=a_{n+1}+a_n+k$, where k is a positive integer.
Find the least k for which a_{1991} and 1991 are not coprime.

3 Let C be a coloring of all edges and diagonals of a convex $n$gon in red and blue (in Romanian, rosu and albastru). Denote by $q_r(C)$ (resp. $q_a(C)$) the number of quadrilaterals having all its

edges and diagonals red (resp. blue).
Prove: $\underset{C}{min}(q_r(C)+q_a(C)) \leq \frac{1}{32}\binom{n}{4}$

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4 Let S be the set of all polygonal areas in a plane. Prove that there is a function $f : S \to (0,1)$ which satisfies $f(S_1 \cup S_2) = f(S_1) + f(S_2)$ for any $S1, S2 \in S$ which have common points only on their borders

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Day 2 June 10th

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5 In a triangle $A_1A_2A_3$, the excribed circles corresponding to sides A_2A_3, A_3A_1, A_1A_2 touch these sides at T_1, T_2, T_3, respectively. If H_1, H_2, H_3 are the orthocenters of triangles $A_1T_2T_3$, $A_2T_3T_1$, $A_3T_1T_2$, respectively, prove that lines H_1T_1, H_2T_2, H_3T_3 are concurrent.

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6 Let $n \geq 3$ be an integer. A finite number of disjoint arcs with the total sum of length $1 - \frac{1}{n}$ are given on a circle of perimeter 1. Prove that there is a regular n-gon whose all vertices lie on the considered arcs

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7 Let $x_1, x_2, ..., x_{2n}$ be positive real numbers with the sum 1. Prove that

$$x_1^2x_2^2...x_n^2 + x_2^2x_3^2...x_{n+1}^2 + ... + x_{2n}^2x_1^2...x_{n-1}^2 < \frac{1}{n^{2n}}$$

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8 Let n, a, b be integers with $n \geq 2$ and $a \notin \{0,1\}$ and let $u(x) = ax + b$ be the function defined on integers. Show that there are infinitely many functions $f : \mathbb{Z} \to \mathbb{Z}$ such that $f_n(x) = \underbrace{f(f(\cdots f}_{n}(x)\cdots)) = u(x)$ for all x.

If $a = 1$, show that there is a b for which there is no f with $f_n(x) \equiv u(x)$.

Day 3

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9 The diagonals of a pentagon $ABCDE$ determine another pentagon $MNPQR$. If $MNPQR$ and $ABCDE$ are similar, must $ABCDE$ be regular?

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10 Let $a_1 < a_2 < \cdots < a_n$ be positive integers. Some colouring of $\mathbb{Z}$ is periodic with period t such that for each $x \in \mathbb{Z}$ exactly one of $x + a_1, x + a_2, \ldots, x + a_n$ is coloured. Prove that $n \mid t$.

Andrei Radulescu-Banu
