Romania Team Selection Test 1996

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Day 1

1 Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function such that for every regular n-gon $A_1A_2\ldots A_n$ we have $f(A_1) + f(A_2) + \cdots + f(A_n) = 0$. Prove that $f(x) = 0$ for all reals x.

2 Find the greatest positive integer n for which there exist n nonnegative integers $x_1, x_2, \ldots, x_n$, not all zero, such that for any $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ from the set $\{-1, 0, 1\}$, not all zero, $\varepsilon_1 x_1 + \varepsilon_2 x_2 + \cdots + \varepsilon_n x_n$ is not divisible by n^3.

3 Let $x, y \in \mathbb{R}$. Show that if the set $A_{x,y} = \{\cos(n\pi x) + \cos(n\pi y) \mid n \in \mathbb{N}\}$ is finite then $x, y \in \mathbb{Q}$.

Vasile Pop

4 Let $ABCD$ be a cyclic quadrilateral and let M be the set of incenters and excenters of the triangles BCD, CDA, DAB, ABC (so 16 points in total). Prove that there exist two sets $\mathcal{K}$ and $\mathcal{L}$ of four parallel lines each, such that every line in $\mathcal{K} \cup \mathcal{L}$ contains exactly four points of M.

Day 2

5 Let A and B be points on a circle $\mathcal{C}$ with center O such that $\angle AOB = \dfrac{\pi}{2}$. Circles $\mathcal{C}_1$ and $\mathcal{C}_2$ are internally tangent to $\mathcal{C}$ at A and B respectively and are also externally tangent to one another. The circle $\mathcal{C}_3$ lies in the interior of $\angle AOB$ and it is tangent externally to $\mathcal{C}_1$, $\mathcal{C}_2$ at P and R and internally tangent to $\mathcal{C}$ at S. Evaluate the value of $\angle PSR$.

7 Let $a \in \mathbb{R}$ and $f_1(x), f_2(x), \ldots, f_n(x) : \mathbb{R} \to \mathbb{R}$ are the additive functions such that for every $x \in \mathbb{R}$ we have $f_1(x)f_2(x)\cdots f_n(x) = ax^n$. Show that there exists $b \in \mathbb{R}$ and $i \in \{1, 2, \ldots, n\}$ such that for every $x \in \mathbb{R}$ we have $f_i(x) = bx$.

8 Let $p_1, p_2, \ldots, p_k$ be the distinct prime divisors of n and let $a_n = \frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_k}$ for $n \geq 2$. Show that for every positive integer $N \geq 2$ the following inequality holds: $\sum_{k=2}^{N} a_2 a_3 \cdots a_k < 1$

Laurentiu Panaitopol

Day 3

9 Let $n \geq 3$ be an integer and let $x_1, x_2, \dots, x_{n-1}$ be nonnegative integers such that

$$\begin{aligned} x_1 + x_2 + \cdots + x_{n-1} &= n \\ x_1 + 2x_2 + \cdots + (n-1)x_{n-1} &= 2n-2. \end{aligned}$$

Find the minimal value of $F(x_1, x_2, \dots, x_n) = \sum_{k=1}^{n-1} k(2n-k)x_k$.

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10 Let n and r be positive integers and A be a set of lattice points in the plane such that any open disc of radius r contains a point of A. Show that
for any coloring of the points of A in n colors there exists four points of the same color which are the vertices of a rectangle.

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11 Find all primes p, q such that $\alpha^{3pq} - \alpha \equiv 0 \pmod{3pq}$ for all integers α.

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12 Let $n \geq 3$ be an integer and let $p \geq 2n-3$ be a prime number. For a set M of n points in the plane, no 3 collinear, let $f : M \to \{0, 1, \dots, p-1\}$ be a function such that

(i) exactly one point of M maps to 0,
(ii) if a circle $\mathcal{C}$ passes through 3 distinct points of $A, B, C \in M$ then $\sum_{P \in M \cap \mathcal{C}} f(P) \equiv 0 \pmod{p}$.

Prove that all the points in M lie on a circle.

Day 4

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13 Let $x_1, x_2, \dots, x_n$ be positive real numbers and $x_{n+1} = x_1 + x_2 + \cdots + x_n$. Prove that

$$\sum_{k=1}^{n} \sqrt{x_k(x_{n+1} - x_k)} \leq \sqrt{\sum_{k=1}^{n} x_{n+1}(x_{n+1} - x_k)}.$$

Mircea Becheanu

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14 Let x, y, z be real numbers. Prove that the following conditions are equivalent:

(i) x, y, z are positive numbers and $\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} \leq 1$;
(ii) $a^2x + b^2y + c^2z > d^2$ holds for every quadrilateral with sides a, b, c, d.

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15 Let S be a set of n concentric circles in the plane. Prove that if a function $f : S \to S$ satisfies the property

$$d(f(A), f(B)) \geq d(A, B)$$

for all $A, B \in S$, then $d(f(A), f(B)) = d(A, B)$, where d is the euclidean distance function.

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16 Let $n \geq 3$ be an integer and let $\mathcal{S} \subset \{1, 2, \ldots, n^3\}$ be a set with $3n^2$ elements. Prove that there exist nine distinct numbers $a_1, a_2, \ldots, a_9 \in \mathcal{S}$ such that the following system has a solution in nonzero integers:

$$\begin{aligned} a_1x + a_2y + a_3z &= 0 \\ a_4x + a_5y + a_6z &= 0 \\ a_7x + a_8y + a_9z &= 0. \end{aligned}$$

Marius Cavachi
