Romania Team Selection Test 1997

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by WakeUp, ehsan2004, hophinhan, Peter, Usetobe, LastPrime, huyhoang, piaulle

Day 1

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1 We are given in the plane a line ℓ and three circles with centres A,B,C such that they are all tangent to ℓ and pairwise externally tangent to each other. Prove that the triangle ABC has an obtuse angle and find all possible values of this this angle.

Mircea Becheanu

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2 Find the number of sets A containing 9 positive integers with the following property: for any positive integer $n\le 500$, there exists a subset $B\subset A$ such that $\sum_{b\in B}b=n$.

Bogdan Enescu & Dan Ismailescu

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3 Let $n\ge 4$ be a positive integer and let M be a set of n points in the plane, where no three points are collinear and not all of the n points being concyclic. Find all real functions $f:M\to\mathbb{R}$ such that for any circle $\mathcal{C}$ containing at least three points from M, the following equality holds:

$$\sum_{P\in\mathcal{C}\cap M} f(P)=0$$

Dorel Mihet

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4 Let ABC be a triangle, D be a point on side BC, and let $\mathcal{O}$ be the circumcircle of triangle ABC. Show that the circles tangent to $\mathcal{O},AD,BD$ and to $\mathcal{O},AD,DC$ are tangent to each other if and only if $\angle BAD=\angle CAD$.

Dan Branzei

Day 2

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1 Let $VA_1A_2\ldots A_n$ be a pyramid, where $n\ge 4$. A plane Π intersects the edges $VA_1,VA_2,\ldots,VA_n$ at the points $B_1,B_2,\ldots,B_n$ respectively such that the polygons $A_1A_2\ldots A_n$ and $B_1B_2\ldots B_n$ are similar. Prove that the plane Π is parallel to the plane containing the base $A_1A_2\ldots A_n$.

Laurentiu Panaitopol

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2 Suppose that A be the set of all positive integer that can write in form a^2+2b^2 (where $a,b\in\mathbb{Z}$ and b is not equal to 0). Show that if p be a prime number and $p^2\in A$ then $p\in A$.

Marcel Tena

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3 Let p be a prime number, $p \geq 5$, and k be a digit in the p-adic representation of positive integers. Find the maximal length of a non constant arithmetic progression whose terms do not contain the digit k in their p-adic representation.

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4 Let p, q, r be distinct prime numbers and let

$$A = \{p^a q^b r^c \mid 0 \leq a, b, c \leq 5\}$$

Find the least $n \in \mathbb{N}$ such that for any $B \subset A$ where $|B| = n$, has elements x and y such that x divides y.

Ioan Tomescu

Day 3

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1 Let $ABCDEF$ be a convex hexagon, and let $P = AB \cap CD$, $Q = CD \cap EF$, $R = EF \cap AB$, $S = BC \cap DE$, $T = DE \cap FA$, $U = FA \cap BC$. Prove that

$\frac{PQ}{CD} = \frac{QR}{EF} = \frac{RP}{AB}$ if and only if $\frac{ST}{DE} = \frac{TU}{FA} = \frac{US}{BC}$

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2 Let P be the set of points in the plane and D the set of lines in the plane. Determine whether there exists a bijective function $f : P \to D$ such that for any three collinear points A, B, C, the lines $f(A)$, $f(B)$, $f(C)$ are either parallel or concurrent.

Gefry Barad

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3 Find all functions $f : \mathbb{R} \to [0; +\infty)$ such that:

$$f(x^2 + y^2) = f(x^2 - y^2) + f(2xy)$$

for all real numbers x and y.

Laurentiu Panaitopol

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4 Let $n \geq 2$ be an integer and let $P(X) = X^n + a_{n-1}X^{n-1} + \ldots + a_1 X + 1$ be a polynomial with positive integer coefficients. Suppose that $a_k = a_{n-k}$ for all $k \in 1, 2, \ldots, n-1$. Prove that there exist infinitely many pairs of positive integers x, y such that $x|P(y)$ and $y|P(x)$.

Remus Nicoara

Day 4

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1 Let $P(X), Q(X)$ be monic irreducible polynomials with rational coefficients. suppose that $P(X)$ and $Q(X)$ have roots α and β respectively, such that $\alpha + \beta$ is rational. Prove that $P(X)^2 - Q(X)^2$ has a rational root.

Bogdan Enescu

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2 Let $a > 1$ be a positive integer. Show that the set of integers

$$\{a^2+a-1, a^3+a^2-1, \ldots, a^{n+1}+a^n-1, \ldots\}$$

contains an infinite subset of pairwise coprime integers.

Mircea Becheanu

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3 The vertices of a regular dodecagon are coloured either blue or red. Find the number of all possible colourings which do not contain monochromatic sub-polygons.

Vasile Pop

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4 Let w be a circle and AB a line not intersecting w. Given a point P_0 on w, define the sequence $P_0, P_1, \ldots$ as follows: P_{n+1} is the second intersection with w of the line passing through B and the second intersection of the line AP_n with w. Prove that for a positive integer k, if $P_0 = P_k$ for some choice of P_0, then $P_0 = P_k$ for any choice of P_0.

Gheorge Eckstein
