Romania Team Selection Test 1998

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Day 1

1 A word of length n is an ordered sequence $x_1x_2\ldots x_n$ where x_i is a letter from the set $\{a,b,c\}$. Denote by A_n the set of words of length n which do not contain any block $x_ix_{i+1}, i=1,2,\ldots,n-1$, of the form aa or bb and by B_n the set of words of length n in which none of the subsequences $x_ix_{i+1}x_{i+2}, i=1,2,\ldots n-2$, contains all the letters a,b,c.
Prove that $|B_{n+1}|=3|A_n|$.

Vasile Pop

2 A parallelepiped has surface area 216 and volume 216. Show that it is a cube.

3 Let $m\geq 2$ be an integer. Find the smallest positive integer $n>m$ such that for any partition with two classes of the set $\{m,m+1,\ldots,n\}$ at least one of these classes contains three numbers a,b,c (not necessarily different) such that $a^b=c$.

Ciprian Manolescu

4 Consider in the plane a finite set of segments such that the sum of their lengths is less than $\sqrt{2}$. Prove that there exists an infinite unit square grid covering the plane such that the lines defining the grid do not intersect any of the segments.

Vasile Pop

Day 2

1 We are given an isosceles triangle ABC such that $BC=a$ and $AB=BC=b$. The variable points $M\in(AC)$ and $N\in(AB)$ satisfy $a^2\cdot AM\cdot AN=b^2\cdot BN\cdot CM$. The straight lines BM and CN intersect in P. Find the locus of the variable point P.

Dan Branzei

2 All the vertices of a convex pentagon are on lattice points. Prove that the area of the pentagon is at least $\frac{5}{2}$.

Bogdan Enescu

3 Find all positive integers (x,n) such that x^n+2^n+1 divides $x^{n+1}+2^{n+1}+1$.

Day 3

1 Let $n \geq 2$ be an integer. Show that there exists a subset $A \in \{1, 2, \dots, n\}$ such that:

i) The number of elements of A is at most $2\lfloor\sqrt{n}\rfloor + 1$

ii) $\{|x-y| \mid x, y \in A, x \neq y\} = \{1, 2, \dots n-1\}$

Radu Todor

2 An infinite arithmetic progression whose terms are positive integers contains the square of an integer and the cube of an integer. Show that it contains the sixth power of an integer.

3 Show that for any positive integer n the polynomial $f(x) = (x^2+x)^{2^n}+1$ cannot be decomposed into the product of two integer non-constant polynomials.

Marius Cavachi

Day 4

1 Let ABC be an equilateral triangle and $n \geq 2$ be an integer. Denote by $\mathcal{A}$ the set of $n-1$ straight lines which are parallel to BC and divide the surface $[ABC]$ into n polygons having the same area and denote by $\mathcal{P}$ the set of $n-1$ straight lines parallel to BC which divide the surface $[ABC]$ into n polygons having the same perimeter.
Prove that the intersection $\mathcal{A} \cap \mathcal{P}$ is empty.

Laurentiu Panaitopol

2 Let $n \geq 3$ be a prime number and $a_1 < a_2 < \cdots < a_n$ be integers. Prove that $a_1, \cdots, a_n$ is an arithmetic progression if and only if there exists a partition of $\{0, 1, 2, \cdots\}$ into sets $A_1, A_2, \cdots, A_n$ such that

$$a_1 + A_1 = a_2 + A_2 = \cdots = a_n + A_n,$$

where $x + A$ denotes the set $\{x + a | a \in A\}$.

3 Let n be a positive integer and $\mathcal{P}_n$ be the set of integer polynomials of the form $a_0 + a_1x + \ldots + a_nx^n$ where $|a_i| \leq 2$ for $i = 0, 1, \ldots, n$. Find, for each positive integer k, the number of elements of the set $A_n(k) = \{f(k) | f \in \mathcal{P}_n\}$.

Marian Andronache

Day 5

1 Find all monotonic functions $u : \mathbb{R} \to \mathbb{R}$ which have the property that there exists a strictly monotonic function $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x+y) = f(x)u(x) + f(y)$$

for all $x, y \in \mathbb{R}$.

Vasile Pop

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2 Find all positive integers k for which the following statement is true: If $F(x)$ is a polynomial with integer coefficients satisfying the condition $0 \leq F(c) \leq k$ for each $c \in \{0, 1, \dots, k+1\}$, then $F(0) = F(1) = \dots = F(k+1)$.

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3 The lateral surface of a cylinder of revolution is divided by $n-1$ planes parallel to the base and m parallel generators into mn cases $(n \geq 1, m \geq 3)$. Two cases will be called neighbouring cases if they have a common side. Prove that it is possible to write a real number in each case such that each number is equal to the sum of the numbers of the neighbouring cases and not all the numbers are zero if and only if there exist integers k, l such that $n+1$ does not divide k and

$$\cos \frac{2l\pi}{m} + \cos \frac{k\pi}{n+1} = \frac{1}{2}$$

Ciprian Manolescu
