Romania Team Selection Test 1999

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Day 1 April 17th

1 a) Prove that it is possible to choose one number out of any 39 consecutive positive integers, having the sum of its digits divisible by 11;

b) Find the first 38 consecutive positive integers none of which have the sum of its digits divisible by 11.

2 Let ABC be an acute triangle. The interior angle bisectors of $\angle ABC$ and $\angle ACB$ meet the opposite sides in L and M respectively. Prove that there is a point K in the interior of the side BC such that the triangle KLM is equilateral if and only if $\angle BAC = 60^\circ$.

3 Prove that for any positive integer n, the number

$$S_n = \binom{2n+1}{0} \cdot 2^{2n} + \binom{2n+1}{2} \cdot 2^{2n-2} \cdot 3 + \cdots + \binom{2n+1}{2n} \cdot 3^n$$

is the sum of two consecutive perfect squares.

Dorin Andrica

4 Show that for all positive real numbers $x_1, x_2, \ldots, x_n$ with product 1, the following inequality holds

$$\frac{1}{n-1+x_1} + \frac{1}{n-1+x_2} + \cdots + \frac{1}{n-1+x_n} \leq 1.$$

Day 2 April 25th

5 Let $x_1, x_2, \ldots, x_n$ be distinct positive integers. Prove that

$$x_1^2 + x_2^2 + \cdots + x_n^2 \geq \frac{2n+1}{3}(x_1 + x_2 + \cdots + x_n).$$

Laurentiu Panaitopol

6 Let ABC be a triangle, H its orthocenter, O its circumcenter, and R its circumradius. Let D be the reflection of the point A across the line BC, let E be the reflection of the point B across the line CA, and let F be the reflection of the point C across the line AB. Prove that the points D, E and F are collinear if and only if $OH = 2R$.

7 Prove that for any integer n, $n \geq 3$, there exist n positive integers $a_1, a_2, \dots, a_n$ in arithmetic progression, and n positive integers in geometric progression $b_1, b_2, \dots, b_n$ such that

$$b_1 < a_1 < b_2 < a_2 < \cdots < b_n < a_n.$$

Give an example of two such progressions having at least five terms.

Mihai Baluna

Day 3 May 15th

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8 Let a be a positive real number and $\{x_n\}_{n\geq 1}$ a sequence of real numbers such that $x_1 = a$ and

$$x_{n+1} \geq (n+2)x_n - \sum_{k=1}^{n-1} kx_k, \ \forall\, n \geq 1.$$

Prove that there exists a positive integer n such that $x_n > 1999!$.

Ciprian Manolescu

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9 Let O, A, B, C be variable points in the plane such that $OA = 4$, $OB = 2\sqrt{3}$ and $OC = \sqrt{22}$. Find the maximum value of the area ABC.

Mihai Baluna

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10 Determine all positive integers n for which there exists an integer m such that $2^n - 1$ is a divisor of $m^2 + 9$.

Day 4 May 22nd

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11 Let a, n be integer numbers, p a prime number such that $p > |a| + 1$. Prove that the polynomial $f(x) = x^n + ax + p$ cannot be represented as a product of two integer polynomials.

Laurentiu Panaitopol

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12 Two circles intersect at two points A and B. A line ℓ which passes through the point A meets the two circles again at the points C and D, respectively. Let M and N be the midpoints of the arcs BC and BD (which do not contain the point A) on the respective circles. Let K be the midpoint of the segment CD. Prove that $\measuredangle MKN = 90^\circ$.

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13 Let $n \geq 3$ and $A_1, A_2, \dots, A_n$ be points on a circle. Find the largest number of acute triangles that can be considered with vertices in these points.

G. Eckstein

Day 5 May 23rd

15 The participants to an international conference are native and foreign scientist. Each native scientist sends a message to a foreign scientist and each foreign scientist sends a message to a native scientist. There are native scientists who did not receive a message.

Prove that there exists a set S of native scientists such that the outer S scientists are exactly those who received messages from those foreign scientists who did not receive messages from scientists belonging to S.

Radu Niculescu

16 Let X be a set with n elements, and let A_1, A_2, ..., A_m be subsets of X such that:

1) $|A_i| = 3$ for every $i \in \{1, 2, ..., m\}$;
2) $|A_i \cap A_j| \le 1$ for all $i, j \in \{1, 2, ..., m\}$ such that $i \neq j$.

Prove that there exists a subset A of X such that A has at least $[\sqrt{2n}]$ elements, and for every $i \in \{1, 2, ..., m\}$, the set A does not contain A_i.

Alternative formulation. Let X be a finite set with n elements and $A_1, A_2, \ldots, A_m$ be three-elements subsets of X, such that $|A_i \cap A_j| \le 1$, for every $i \neq j$. Prove that there exists $A \subseteq X$ with $|A| \ge \lfloor \sqrt{2n} \rfloor$, such that none of A_i's is a subset of A.

17 A polyhedron P is given in space. Find whether there exist three edges in P which can be the sides of a triangle. Justify your answer!

Barbu Berceanu