Romania Team Selection Test 2000

www.artofproblemsolving.com/community/c4455

by WakeUp, manlio, Max D.R., Amir Hossein, orl, Valentin Vornicu, CoBa_c_Kacka, Agr_94_Math, Cezar Lupu

Day 1

---

1 Let $n \geq 2$ be a positive integer. Find the number of functions $f : \{1, 2, \ldots, n\} \to \{1, 2, 3, 4, 5\}$ which have the following property: $|f(k+1) - f(k)| \geq 3$, for any $k = 1, 2, \ldots n-1$.

Vasile Pop

---

2 Let $n \geq 1$ be a positive integer and $x_1, x_2 \ldots, x_n$ be real numbers such that $|x_{k+1} - x_k| \leq 1$ for $k = 1, 2, \ldots, n-1$. Prove that

$$\sum_{k=1}^{n} |x_k| - \left|\sum_{k=1}^{n} x_k\right| \leq \frac{n^2-1}{4}$$

Gh. Eckstein

---

3 Prove that for any positive integers n and k there exist positive integers $a > b > c > d > e > k$ such that

$$n = \binom{a}{3} \pm \binom{b}{3} \pm \binom{c}{3} \pm \binom{d}{3} \pm \binom{e}{3}$$

Radu Ignat

---

4 Let $P_1P_2 \ldots P_n$ be a convex polygon in the plane. We assume that for any arbitrary choice of vertices P_i, P_j there exists a vertex in the polygon P_k distinct from P_i, P_j such that $\angle P_iP_kP_j = 60^\circ$. Show that $n = 3$.

Radu Todor

Day 2

---

1 Prove that the equation $x^3 + y^3 + z^3 = t^4$ has infinitely many solutions in positive integers such that $\gcd(x, y, z, t) = 1$.

Mihai Pitticari & Sorin Rdulescu

---

2 Let ABC be a triangle and M be an interior point. Prove that

$$\min\{MA, MB, MC\} + MA + MB + MC < AB + AC + BC.$$

3 Determine all pairs (m,n) of positive integers such that a $m \times n$ rectangle can be tiled with L-trominoes.

Day 3

1 Let $a > 1$ be an odd positive integer. Find the least positive integer n such that 2^{2000} is a divisor of $a^n - 1$.

Mircea Becheanu

2 Let ABC be an acute-angled triangle and M be the midpoint of the side BC. Let N be a point in the interior of the triangle ABC such that $\angle NBA = \angle BAM$ and $\angle NCA = \angle CAM$. Prove that $\angle NAB = \angle MAC$.

Gabriel Nagy

3 Let S be the set of interior points of a sphere and C be the set of interior points of a circle. Find, with proof, whether there exists a function $f : S \to C$ such that $d(A,B) \leq d(f(A), f(B))$ for any two points $A, B \in S$ where $d(X,Y)$ denotes the distance between the points X and Y.

Marius Cavachi

Day 4

1 Let P_1 be a regular n-gon, where $n \in \mathbb{N}$. We construct P_2 as the regular n-gon whose vertices are the midpoints of the edges of P_1. Continuing analogously, we obtain regular n-gons $P_3, P_4, \ldots, P_m$. For $m \geq n^2 - n + 1$, find the maximum number k such that for any colouring of vertices of $P_1, \ldots, P_m$ in k colours there exists an isosceles trapezium $ABCD$ whose vertices A, B, C, D have the same colour.

Radu Ignat

2 Let P, Q be two monic polynomials with complex coefficients such that $P(P(x)) = Q(Q(x))$ for all x. Prove that $P = Q$.

Marius Cavachi

3 Prove that every positive rational number can be represented in the form $\dfrac{a^3 + b^3}{c^3 + d^3}$ where a,b,c,d are positive integers.