Romania Team Selection Test 2001

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Day 1

1 Show that if a,b,c are complex numbers that such that

$$(a+b)(a+c)=b \qquad (b+c)(b+a)=c \qquad (c+a)(c+b)=a$$

then a,b,c are real numbers.

2 a) Let $f,g:\mathbb{Z}\rightarrow\mathbb{Z}$ be one to one maps. Show that the function $h:\mathbb{Z}\rightarrow\mathbb{Z}$ defined by $h(x)=f(x)g(x)$, for all $x\in\mathbb{Z}$, cannot be a surjective function.

b) Let $f:\mathbb{Z}\rightarrow\mathbb{Z}$ be a surjective function. Show that there exist surjective functions $g,h:\mathbb{Z}\rightarrow\mathbb{Z}$ such that $f(x)=g(x)h(x)$, for all $x\in\mathbb{Z}$.

3 The sides of a triangle have lengths a,b,c. Prove that:

$$(-a+b+c)(a-b+c)+(a-b+c)(a+b-c)+(a+b-c)(-a+b+c)$$
$$\le\sqrt{abc}(\sqrt{a}+\sqrt{b}+\sqrt{c})$$

4 Three schools have 200 students each. Every student has at least one friend in each school (if the student a is a friend of the student b then b is a friend of a).
It is known that there exists a set E of 300 students (among the 600) such that for any school S and any two students $x,y\in E$ but not in S, the number of friends in S of x and y are different.
Show that one can find a student in each school such that they are friends with each other.

Day 2

1 Find all polynomials with real coefficients P such that

$$P(x)P(2x^2-1)=P(x^2)P(2x-1)$$

for every $x\in\mathbb{R}$.

2 The vertices A,B,C and D of a square lie outside a circle centred at M. Let AA',BB',CC',DD' be tangents to the circle. Assume that the segments AA',BB',CC',DD' are the consecutive sides of a quadrilateral p in which a circle is inscribed. Prove that p has an axis of symmetry.

3 Find the least $n \in N$ such that among any n rays in space sharing a common origin there exist two which form an acute angle.

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4 Show that the set of positive integers that cannot be represented as a sum of distinct perfect squares is finite.

Day 3

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1 Let n be a positive integer and $f(x) = a_m x^m + \ldots + a_1 X + a_0$, with $m \geq 2$, a polynomial with integer coefficients such that:
a) $a_2, a_3 \ldots a_m$ are divisible by all prime factors of n,
b) a_1 and n are relatively prime.
Prove that for any positive integer k, there exists a positive integer c, such that $f(c)$ is divisible by n^k.

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3 Let p and q be relatively prime positive integers. A subset S of $\{0, 1, 2, \ldots\}$ is called **ideal** if $0 \in S$ and for each element $n \in S$, the integers $n + p$ and $n + q$ belong to S. Determine the number of ideal subsets of $\{0, 1, 2, \ldots\}$.

Day 4

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1 Find all pairs (m, n) of positive integers, with $m, n \geq 2$, such that $a^n - 1$ is divisible by m for each $a \in \{1, 2, 3, \ldots, n\}$.

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2 Prove that there is no function $f : (0, \infty) \to (0, \infty)$ such that

$$f(x + y) \geq f(x) + yf(f(x))$$

for every $x, y \in (0, \infty)$.

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3 The tangents at A and B to the circumcircle of the acute triangle ABC intersect the tangent at C at the points D and E, respectively. The line AE intersects BC at P and the line BD intersects AC at R. Let Q and S be the midpoints of the segments AP and BR respectively. Prove that $\angle ABQ = \angle BAS$.

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4 Consider a convex polyhedron P with vertices $V_1, \ldots, V_p$. The distinct vertices V_i and V_j are called *neighbours* if they belong to the same face of the polyhedron. To each vertex V_k we assign a number $v_k(0)$, and construct inductively the sequence $v_k(n)$ $(n \geq 0)$ as follows: $v_k(n+1)$ is the average of the $v_j(n)$ for all neighbours V_j of V_k . If all numbers $v_k(n)$ are integers, prove that there exists the positive integer N such that all $v_k(n)$ are equal for $n \geq N$.
