Romania Team Selection Test 2002

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Day 1

1 Find all sets A and B that satisfy the following conditions:

a) $A \cup B = \mathbb{Z}$;

b) if $x \in A$ then $x-1 \in B$;

c) if $x, y \in B$ then $x+y \in A$.

Laurentiu Panaitopol

2 The sequence (a_n) is defined by: $a_0 = a_1 = 1$ and $a_{n+1} = 14a_n - a_{n-1}$ for all $n \geq 1$.
Prove that $2a_n - 1$ is a perfect square for any $n \geq 0$.

3 Let M and N be the midpoints of the respective sides AB and AC of an acute-angled triangle ABC. Let P be the foot of the perpendicular from N onto BC and let A_1 be the midpoint of MP. Points B_1 and C_1 are obtained similarly. If AA_1, BB_1 and CC_1 are concurrent, show that the triangle ABC is isosceles.

Mircea Becheanu

4 For any positive integer n, let $f(n)$ be the number of possible choices of signs $+$ or $-$ in the algebraic expression $\pm 1 \pm 2 \ldots \pm n$, such that the obtained sum is zero. Show that $f(n)$ satisfies the following conditions:
a) $f(n) = 0$ for $n = 1 \pmod 4$ or $n = 2 \pmod 4$.
b) $2^{\frac{n}{2}-1} \leq f(n) \leq 2^n - 2^{\lfloor \frac{n}{2} \rfloor + 1}$, for $n = 0 \pmod 4$ or $n = 3 \pmod 4$.

Ioan Tomsecu

Day 2

1 Let $ABCD$ be a unit square. For any interior points M, N such that the line MN does not contain a vertex of the square, we denote by $s(M, N)$ the least area of the triangles having their vertices in the set of points $\{A, B, C, D, M, N\}$. Find the least number k such that $s(M, N) \leq k$, for all points M, N.

Dinu erbnescu

2 Let $P(x)$ and $Q(x)$ be integer polynomials of degree p and q respectively. Assume that $P(x)$ divides $Q(x)$ and all their coefficients are either 1 or 2002. Show that $p+1$ is a divisor of $q+1$.

Mihai Cipu

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3 Let a,b be positive real numbers. For any positive integer n, denote by x_n the sum of digits of the number $[an+b]$ in it's decimal representation. Show that the sequence $(x_n)_{n\geq 1}$ contains a constant subsequence.

Laurentiu Panaitopol

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4 At an international conference there are four official languages. Any two participants can speak in one of these languages. Show that at least 60% of the participants can speak the same language.

Mihai Baluna

Day 3

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1 Let $ABCDE$ be a cyclic pentagon inscribed in a circle of centre O which has angles $\angle B = 120^\circ, \angle C = 120^\circ, \angle D = 130^\circ, \angle E = 100^\circ$. Show that the diagonals BD and CE meet at a point belonging to the diameter AO.

Dinu erbnescu

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2 Let $n\geq 4$ be an integer, and let $a_1,a_2,\ldots,a_n$ be positive real numbers such that

$$a_1^2+a_2^2+\cdots+a_n^2=1.$$

Prove that the following inequality takes place

$$\frac{a_1}{a_2^2+1}+\cdots+\frac{a_n}{a_1^2+1}\geq \frac{4}{5}\left(a_1\sqrt{a_1}+\cdots+a_n\sqrt{a_n}\right)^2.$$

Bogdan Enescu, Mircea Becheanu

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3 Let n be a positive integer. S is the set of nonnegative integers a such that $1<a<n$ and $a^{a-1}-1$ is divisible by n. Prove that if $S=\{n-1\}$ then $n=2p$ where p is a prime number.

Mihai Cipu and Nicolae Ciprian Bonciocat

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4 Let $f:\mathbb{Z}\to\{1,2,\ldots,n\}$ be a function such that $f(x)\neq f(y)$, for all $x,y\in\mathbb{Z}$ such that $|x-y|\in\{2,3,5\}$. Prove that $n\geq 4$.

Ioan Tomescu

Day 4

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1 Let $(a_n)_{n\geq 1}$ be a sequence of positive integers defined as $a_1, a_2 > 0$ and a_{n+1} is the least prime divisor of $a_{n-1} + a_n$, for all $n \geq 2$.

Prove that a real number x whose decimals are digits of the numbers $a_1, a_2, \dots a_n, \dots$ written in order, is a rational number.

Laurentiu Panaitopol

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2 Find the least positive real number r with the following property:

Whatever four disks are considered, each with centre on the edges of a unit square and the sum of their radii equals r, there exists an equilateral triangle which has its edges in three of the disks.

Radu Gologan

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3 After elections, every parliament member (PM), has his own absolute rating. When the parliament set up, he enters in a group and gets a relative rating. The relative rating is the ratio of its own absolute rating to the sum of all absolute ratings of the PMs in the group. A PM can move from one group to another only if in his new group his relative rating is greater. In a given day, only one PM can change the group. Show that only a finite number of group moves is possible.

(A rating is positive real number.)

Day 5

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1 Let m, n be positive integers of distinct parities and such that $m < n < 5m$. Show that there exists a partition with two element subsets of the set $\{1, 2, 3, \dots, 4mn\}$ such that the sum of numbers in each set is a perfect square.

Dinu erbnescu

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2 Let ABC be a triangle such that $AC \neq BC, AB < AC$ and let K be it's circumcircle. The tangent to K at the point A intersects the line BC at the point D. Let K_1 be the circle tangent to K and to the segments $(AD), (BD)$. We denote by M the point where K_1 touches (BD). Show that $AC = MC$ if and only if AM is the bisector of the $\angle DAB$.

Neculai Roman

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3 There are n players, $n \geq 2$, which are playing a card game with np cards in p rounds. The cards are coloured in n colours and each colour is labelled with the numbers $1, 2, \dots, p$. The game submits to the following rules:
each player receives p cards.
the player who begins the first round throws a card and each player has to discard a card of the same colour, if he has one; otherwise they can give an arbitrary card.
the winner of the round is the player who has put the greatest card of the same colour as the

first one.
the winner of the round starts the next round with a card that he selects and the play continues with the same rules.
the played cards are out of the game.
Show that if all cards labelled with number 1 are winners, then $p \geq 2n$.

Barbu Berceanu
