Romania Team Selection Test 2005

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by Valentin Vornicu, grobber

Day 1 March 31st

1 Solve the equation $3^x = 2^x y + 1$ in positive integers.

2 Let $n \geq 1$ be an integer and let X be a set of n^2+1 positive integers such that in any subset of X with $n+1$ elements there exist two elements $x \neq y$ such that $x \mid y$. Prove that there exists a subset $\{x_1, x_2, \ldots, x_{n+1}\} \in X$ such that $x_i \mid x_{i+1}$ for all $i = 1, 2, \ldots, n$.

3 Prove that if the distance from a point inside a convex polyhedra with n faces to the vertices of the polyhedra is at most 1, then the sum of the distances from this point to the faces of the polyhedra is smaller than $n-2$.

Calin Popescu

Day 2 April 1st

1 Prove that in any convex polygon with $4n+2$ sides ($n \geq 1$) there exist two consecutive sides which form a triangle of area at most $\frac{1}{6n}$ of the area of the polygon.

2 Let m, n be co-prime integers, such that m is even and n is odd. Prove that the following expression does not depend on the values of m and n:

$$\frac{1}{2n} + \sum_{k=1}^{n-1} (-1)^{\left[\frac{mk}{n}\right]} \left\{ \frac{mk}{n} \right\}.$$

Bogdan Enescu

3 A sequence of real numbers $\{a_n\}_n$ is called a *bs* sequence if $a_n = |a_{n+1} - a_{n+2}|$, for all $n \geq 0$. Prove that a bs sequence is bounded if and only if the function f given by $f(n,k) = a_n a_k (a_n - a_k)$, for all $n, k \geq 0$ is the null function.

Mihai Baluna - ISL 2004

Day 3 April 19th

1 Let $A_0A_1A_2A_3A_4A_5$ be a convex hexagon inscribed in a circle. Define the points A'_0, A'_2, A'_4 on the circle, such that

$$A_0A'_0 \parallel A_2A_4, \quad A_2A'_2 \parallel A_4A_0, \quad A_4A'_4 \parallel A_2A_0.$$

Let the lines A'_0A_3 and A_2A_4 intersect in A'_3, the lines A'_2A_5 and A_0A_4 intersect in A'_5 and the lines A'_4A_1 and A_0A_2 intersect in A'_1.

Prove that if the lines A_0A_3, A_1A_4 and A_2A_5 are concurrent then the lines $A_0A'_3$, $A_4A'_1$ and $A_2A'_5$ are also concurrent.

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2 Let ABC be a triangle, and let D, E, F be 3 points on the sides BC, CA and AB respectively, such that the inradii of the triangles AEF, BDF and CDE are equal with half of the inradius of the triangle ABC. Prove that D, E, F are the midpoints of the sides of the triangle ABC.

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3 Let P be a polygon (not necessarily convex) with n vertices, such that all its sides and diagonals are less or equal with 1 in length. Prove that the area of the polygon is less than $\dfrac{\sqrt{3}}{2}$.

Day 4 May 23rd

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1 Let $a \in \mathbb{R} - \{0\}$. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(a + x) = f(x) - x$ for all $x \in \mathbb{R}$.

Dan Schwartz

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2 On the edges of a convex polyhedra we draw arrows such that from each vertex at least an arrow is pointing in and at least one is pointing out.
Prove that there exists a face of the polyhedra such that the arrows on its edges form a circuit.

Dan Schwartz

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3 Let $n \geq 0$ be an integer and let $p \equiv 7 \pmod 8$ be a prime number. Prove that

$$\sum_{k=1}^{p-1} \left\{ \frac{k^{2^n}}{p} - \frac{1}{2} \right\} = \frac{p-1}{2}.$$

Clin Popescu

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4 a) Prove that there exists a sequence of digits $\{c_n\}_{n\geq 1}$ such that or each $n \geq 1$ no matter how we interlace k_n digits, $1 \leq k_n \leq 9$, between c_n and c_{n+1}, the infinite sequence thus obtained does not represent the fractional part of a rational number.

b) Prove that for $1 \leq k_n \leq 10$ there is no such sequence $\{c_n\}_{n\geq 1}$.

Dan Schwartz

Day 5 May 24th

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1 On a 2004×2004 chess table there are 2004 queens such that no two are attacking each other[1].

Prove that there exist two queens such that in the rectangle in which the center of the squares on which the queens lie are two opposite corners, has a semiperimeter of 2004.

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2 Let $n \geq 2$ be an integer. Find the smallest real value $\rho(n)$ such that for any $x_i > 0$, $i = 1, 2, \ldots, n$ with $x_1 x_2 \cdots x_n = 1$, the inequality

$$\sum_{i=1}^{n} \frac{1}{x_i} \leq \sum_{i=1}^{n} x_i^r$$

is true for all $r \geq \rho(n)$.

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3 Let $\mathbb{N} = \{1, 2, \ldots\}$. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that for all $m, n \in \mathbb{N}$ the number $f^2(m) + f(n)$ is a divisor of $(m^2 + n)^2$.

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4 We consider a polyhedra which has exactly two vertices adjacent with an odd number of edges, and these two vertices are lying on the same edge.

Prove that for all integers $n \geq 3$ there exists a face of the polyhedra with a number of sides not divisible by n.

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[1] two queens attack each other if they lie on the same row, column or direction parallel with on of the main diagonals of the table