Romania Team Selection Test 2006

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Day 1 April 19th

1 Let ABC and AMN be two similar triangles with the same orientation, such that $AB=AC$, $AM=AN$ and having disjoint interiors. Let O be the circumcenter of the triangle MAB. Prove that the points O, C, N, A lie on the same circle if and only if the triangle ABC is equilateral.

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2 Let p a prime number, $p\geq 5$. Find the number of polynomials of the form

$$x^p + px^k + px^l + 1, \quad k > l, \quad k, l \in \left\{1,2,\dots,p-1\right\},$$

which are irreducible in $\mathbb{Z}[X]$.

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4 The real numbers $a_1, a_2, \dots, a_n$ are given such that $|a_i| \leq 1$
for all $i=1,2,\dots,n$ and $a_1+a_2+\cdots+a_n=0$.

a) Prove that there exists $k\in\{1,2,\dots,n\}$ such that

$$|a_1+2a_2+\cdots+ka_k| \leq \frac{2k+1}{4}.$$

b) Prove that for $n>2$ the bound above is the best possible.

Radu Gologan, Dan Schwarz

Day 2 April 20th

1 Let $\{a_n\}_{n\geq 1}$ be a sequence with $a_1=1$, $a_2=4$ and for all $n>1$,

$$a_n = \sqrt{a_{n-1}a_{n+1}+1}.$$

a) Prove that all the terms of the sequence are positive integers.

b) Prove that $2a_na_{n+1}+1$ is a perfect square for all positive integers n.

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2 Let ABC be a triangle with $\angle B = 30^\circ$. We consider the closed disks of radius $\frac{AC}{3}$, centered in A, B, C. Does there exist an equilateral triangle with one vertex in each of the 3 disks?

Radu Gologan, Dan Schwarz

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3 For which pairs of positive integers (m,n) there exists a set A such that for all positive integers x,y, if $|x-y|=m$, then at least one of the numbers x,y belongs to the set A, and if $|x-y|=n$, then at least one of the numbers x,y does not belong to the set A?

Adapted by Dan Schwarz from A.M.M.

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4 Let x_i, $1\leq i\leq n$ be real numbers. Prove that

$$\sum_{1\leq i<j\leq n} |x_i+x_j| \geq \frac{n-2}{2}\sum_{i=1}^{n}|x_i|.$$

Discrete version by Dan Schwarz of a Putnam problem

Day 3 May 16th

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1 The circle of center I is inscribed in the convex quadrilateral $ABCD$. Let M and N be points on the segments AI and CI, respectively, such that $\angle MBN = \frac{1}{2}\angle ABC$. Prove that $\angle MDN = \frac{1}{2}\angle ADC$.

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2 Let A be point in the exterior of the circle $\mathcal{C}$. Two lines passing through A intersect the circle $\mathcal{C}$ in points B and C (with B between A and C) respectively in D and E (with D between A and E). The parallel from D to BC intersects the second time the circle $\mathcal{C}$ in F. Let G be the second point of intersection between the circle $\mathcal{C}$ and the line AF and M the point in which the lines AB and EG intersect. Prove that

$$\frac{1}{AM} = \frac{1}{AB} + \frac{1}{AC}.$$

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3 Let γ be the incircle in the triangle $A_0A_1A_2$. For all $i\in\{0,1,2\}$ we make the following constructions (all indices are considered modulo 3): γ_i is the circle tangent to γ which passes through the points A_{i+1} and A_{i+2}; T_i is the point of tangency between γ_i and γ; finally, the common tangent in T_i of γ_i and γ intersects the line $A_{i+1}A_{i+2}$ in the point P_i. Prove that

a) the points P_0, P_1 and P_2 are collinear;

b) the lines A_0T_0, A_1T_1 and A_2T_2 are concurrent.

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4 Let a,b,c be positive real numbers such that $a+b+c=3$. Prove that:

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2} \geq a^2+b^2+c^2.$$

Day 4 May 19th

1 Let r and s be two rational numbers. Find all functions $f : \mathbb{Q} \to \mathbb{Q}$ such that for all $x, y \in \mathbb{Q}$ we have

$$f(x + f(y)) = f(x + r) + y + s.$$

2 Find all non-negative integers m, n, p, q such that

$$p^m q^n = (p + q)^2 + 1.$$

3 Let $n > 1$ be an integer. A set $S \subset \{0, 1, 2, \dots, 4n-1\}$ is called *rare* if, for any $k \in \{0, 1, \dots, n-1\}$, the following two conditions take place at the same time

(1) the set $S \cap \{4k - 2, 4k - 1, 4k, 4k + 1, 4k + 2\}$ has at most two elements;

(2) the set $S \cap \{4k + 1, 4k + 2, 4k + 3\}$ has at most one element.

Prove that the set $\{0, 1, 2, \dots, 4n - 1\}$ has exactly $8 \cdot 7^{n-1}$ rare subsets.

4 Let p, q be two integers, $q \geq p \geq 0$. Let $n \geq 2$ be an integer and $a_0 = 0, a_1 \geq 0, a_2, \dots, a_{n-1}, a_n = 1$ be real numbers such that

$$a_k \leq \frac{a_{k-1} + a_{k+1}}{2}, \ \forall\, k = 1, 2, \dots, n - 1.$$

Prove that

$$(p + 1) \sum_{k=1}^{n-1} a_k^p \geq (q + 1) \sum_{k=1}^{n-1} a_k^q.$$

Day 5 May 20th

1 Let n be a positive integer of the form $4k + 1$, $k \in \mathbb{N}$ and $A = \{a^2 + nb^2 \mid a, b \in \mathbb{Z}\}$. Prove that there exist integers x, y such that $x^n + y^n \in A$ and $x + y \notin A$.

2 Let m and n be positive integers and S be a subset with $(2^m - 1)n + 1$ elements of the set $\{1, 2, 3, \dots, 2^m n\}$. Prove that S contains $m+1$ distinct numbers $a_0, a_1, \dots, a_m$ such that $a_{k-1} \mid a_k$ for all $k = 1, 2, \dots, m$.

3 Let $x_1 = 1$, x_2, x_3, $\ldots$ be a sequence of real numbers such that for all $n \geq 1$ we have

$$x_{n+1} = x_n + \frac{1}{2x_n}.$$

Prove that

$$\lfloor 25x_{625} \rfloor = 625.$$

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4 Let ABC be an acute triangle with $AB \neq AC$. Let D be the foot of the altitude from A and ω the circumcircle of the triangle. Let ω_1 be the circle tangent to AD, BD and ω. Let ω_2 be the circle tangent to AD, CD and ω. Let ℓ be the interior common tangent to both ω_1 and ω_2, different from AD.

Prove that ℓ passes through the midpoint of BC if and only if $2BC = AB + AC$.
