Romania Team Selection Test 2007

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Day 1 April 13th

1 If $a_1, a_2, \dots, a_n \geq 0$ are such that
$$a_1^2 + \cdots + a_n^2 = 1,$$
then find the maximum value of the product $(1-a_1)\cdots(1-a_n)$.

2 Let $f : \mathbb{Q} \to \mathbb{R}$ be a function such that
$$|f(x) - f(y)| \leq (x-y)^2$$
for all $x, y \in \mathbb{Q}$. Prove that f is constant.

3 Let $A_1A_2 \dots A_{2n}$ be a convex polygon and let P be a point in its interior such that it doesn't lie on any of the diagonals of the polygon. Prove that there is a side of the polygon such that none of the lines $PA_1, \dots, PA_{2n}$ intersects it in its interior.

4 Let $\mathcal{O}_1$ and $\mathcal{O}_2$ two exterior circles. Let A, B, C be points on $\mathcal{O}_1$ and D, E, F points on $\mathcal{O}_1$ such that AD and BE are the common exterior tangents to these two circles and CF is one of the interior tangents to these two circles, and such that C, F are in the interior of the quadrilateral $ABED$. If $CO_1 \cap AB = \{M\}$ and $FO_2 \cap DE = \{N\}$ then prove that MN passes through the middle of CF.

Day 2 April 14th

1 Let
$$f = X^n + a_{n-1}X^{n-1} + \ldots + a_1X + a_0$$
be an integer polynomial of degree $n \geq 3$ such that $a_k + a_{n-k}$ is even for all $k \in \overline{1, n-1}$ and a_0 is even.
Suppose that $f = gh$, where g, h are integer polynomials and $\deg g \leq \deg h$ and all the coefficients of h are odd.
Prove that f has an integer root.

2 Let ABC be a triangle, E and F the points where the incircle and A-excircle touch AB, and D the point on BC such that the triangles ABD and ACD have equal in-radii. The lines DB and DE intersect the circumcircle of triangle ADF again in the points X and Y.

Prove that $XY \parallel AB$ if and only if $AB = AC$.

3 Find all subsets A of $\{1,2,3,4,\ldots\}$, with $|A| \geq 2$, such that for all $x,y \in A,\ x \neq y$, we have that $\frac{x+y}{\gcd(x,y)} \in A$.

Dan Schwarz

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4 Let S be the set of n-uples $(x_1, x_2, \ldots, x_n)$ such that $x_i \in \{0,1\}$ for all $i \in \overline{1,n}$, where $n \geq 3$. Let $M(n)$ be the smallest integer with the property that any subset of S with at least $M(n)$ elements contains at least three n-uples

$$(x_1, \ldots, x_n),\ (y_1, \ldots, y_n),\ (z_1, \ldots, z_n)$$

such that

$$\sum_{i=1}^{n} (x_i - y_i)^2 = \sum_{i=1}^{n} (y_i - z_i)^2 = \sum_{i=1}^{n} (z_i - x_i)^2 .$$

(a) Prove that $M(n) \leq \left\lfloor \frac{2^{n+1}}{n} \right\rfloor + 1$.
(b) Compute $M(3)$ and $M(4)$.

Day 3

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1 Let $\mathcal{F}$ be the set of all the functions $f : \mathcal{P}(S) \longrightarrow \mathbb{R}$ such that for all $X, Y \subseteq S$, we have $f(X \cap Y) = \min(f(X), f(Y))$, where S is a finite set (and $\mathcal{P}(S)$ is the set of its subsets). Find

$$\max_{f \in \mathcal{F}} |\mathrm{Im}(f)|.$$

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2 Prove that for n, p integers, $n \geq 4$ and $p \geq 4$, the proposition $\mathcal{P}(n,p)$

$$\sum_{i=1}^{n} \frac{1}{{x_i}^p} \geq \sum_{i=1}^{n} {x_i}^p \quad \text{for} \quad x_i \in \mathbb{R}, \quad x_i > 0, \quad i = 1, \ldots, n\ , \quad \sum_{i=1}^{n} x_i = n,$$

is false.

Dan Schwarz

In the competition, the students were informed (fact that doesn't actually relate to the problem's solution) that the propositions $\mathcal{P}(4,3)$ are $\mathcal{P}(3,4)$ true.

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3 Let a_i, $i = 1, 2, \ldots, n$, $n \geq 3$, be positive integers, having the greatest common divisor 1, such that

$$a_j \text{ divide } \sum_{i=1}^{n} a_i$$

for all $j = 1, 2, \dots, n$. Prove that

$$\prod_{i=1}^{n} a_i \text{ divides } \left(\sum_{i=1}^{n} a_i\right)^{n-2}.$$

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4 The points M, N, P are chosen on the sides BC, CA, AB of a triangle ΔABC, such that the triangle ΔMNP is acute-angled. We denote with x the length of the shortest altitude of the triangle ΔABC, and with X the length of the longest altitudes of the triangle ΔMNP. Prove that $x \le 2X$.

Day 4

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1 Prove that the function $f : \mathbb{N} \longrightarrow \mathbb{Z}$ defined by $f(n) = n^{2007} - n!$, is injective.

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2 Let $A_1A_2A_3A_4A_5$ be a convex pentagon, such that

$$[A_1A_2A_3] = [A_2A_3A_4] = [A_3A_4A_5] = [A_4A_5A_1] = [A_5A_1A_2].$$

Prove that there exists a point M in the plane of the pentagon such that

$$[A_1MA_2] = [A_2MA_3] = [A_3MA_4] = [A_4MA_5] = [A_5MA_1].$$

Here $[XYZ]$ stands for the area of the triangle ΔXYZ.

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3 Consider the set $E = \{1, 2, \dots, 2n\}$. Prove that an element $c \in E$ can belong to a subset $A \subset E$, having n elements, and such that any two distinct elements in A do not divide one each other, if and only if

$$c > n\left(\frac{2}{3}\right)^{k+1},$$

where k is the exponent of 2 in the factoring of c.

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4 i) Find all infinite arithmetic progressions formed with positive integers such that there exists a number $N \in \mathbb{N}$, such that for any prime p, $p > N$,
the p-th term of the progression is also prime.

ii) Find all polynomials $f(X) \in \mathbb{Z}[X]$, such that there exist $N \in \mathbb{N}$, such that for any prime p, $p > N$, $|f(p)|$ is also prime.

Dan Schwarz

Day 5

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1 In a circle with center O is inscribed a polygon, which is triangulated. Show that the sum of the squares of the distances from O to the incenters of the formed triangles is independent of the triangulation.

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2 Let ABC be a triangle, and ω_a, ω_b, ω_c be circles inside ABC, that are tangent (externally) one to each other, such that ω_a is tangent to AB and AC, ω_b is tangent to BA and BC, and ω_c is tangent to CA and CB. Let D be the common point of ω_b and ω_c, E the common point of ω_c and ω_a, and F the common point of ω_a and ω_b. Show that the lines AD, BE and CF have a common point.

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3 Let $ABCDE$ be a convex pentagon, such that $AB = BC, CD = DE, \angle B + \angle D = 180^{\circ}$, and it's area is $\sqrt{2}$.

a) If $\angle B = 135^{\circ}$, find the length of $[BD]$.

b) Find the minimum of the length of $[BD]$.

Day 6

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1 Let $ABCD$ be a parallelogram with no angle equal to 60°. Find all pairs of points E, F, in the plane of $ABCD$, such that triangles AEB and BFC are isosceles, of basis AB, respectively BC, and triangle DEF is equilateral.

Valentin Vornicu

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2 The world-renowned Marxist theorist *Joric* is obsessed with both mathematics and social egalitarianism. Therefore, for any decimal representation of a positive integer n, he tries to partition its digits into two groups, such that the difference between the sums of the digits in each group be as small as possible. Joric calls this difference the *defect* of the number n. Determine the average value of the defect (over all positive integers), that is, if we denote by $\delta(n)$ the defect of n, compute

$$\lim_{n\to\infty}\frac{\sum_{k=1}^{n}\delta(k)}{n}.$$

Iurie Boreico

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3 Three travel companies provide transportation between n cities, such that each connection between a pair of cities is covered by one company only. Prove that, for $n \geq 11$, there must exist a round-trip through some four cities, using the services of a same company, while for $n < 11$ this is not anymore necessarily true.

Dan Schwarz

Day 7

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1 For $n \in \mathbb{N}$, $n \geq 2$, $a_i, b_i \in \mathbb{R}$, $1 \leq i \leq n$, such that

$$\sum_{i=1}^{n} a_i^2 = \sum_{i=1}^{n} b_i^2 = 1, \sum_{i=1}^{n} a_i b_i = 0.$$

Prove that

$$\left(\sum_{i=1}^{n} a_i\right)^2 + \left(\sum_{i=1}^{n} b_i\right)^2 \leq n.$$

Cezar Lupu & Tudorel Lupu

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2 Let ABC be a triangle, let E, F be the tangency points of the incircle $\Gamma(I)$ to the sides AC, respectively AB, and let M be the midpoint of the side BC. Let $N = AM \cap EF$, let $\gamma(M)$ be the circle of diameter BC, and let X, Y be the other (than B, C) intersection points of BI, respectively CI, with γ. Prove that

$$\frac{NX}{NY} = \frac{AC}{AB}.$$

Cosmin Pohoata

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3 The problem is about real polynomial functions, denoted by f, of degree $\deg f$.

a) Prove that a polynomial function f can't be wrriten as sum of at most $\deg f$ periodic functions.

b) Show that if a polynomial function of degree 1 is written as sum of two periodic functions, then they are unbounded on every interval (thus, they are "wild").

c) Show that every polynomial function of degree 1 can be written as sum of two periodic functions.

d) Show that every polynomial function f can be written as sum of $\deg f + 1$ periodic functions.

e) Give an example of a function that can't be written as a finite sum of periodic functions.

Dan Schwarz
