Romania Team Selection Test 2012

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by Drytime

Day 1

1 Let $n_1, \dots, n_k$ be positive integers, and define $d_1 = 1$ and $d_i = \frac{(n_1, \dots, n_{i-1})}{(n_1, \dots, n_i)}$, for $i \in \{2, \dots, k\}$, where $(m_1, \dots, m_\ell)$ denotes the greatest common divisor of the integers $m_1, \dots, m_\ell$. Prove that the sums

$$\sum_{i=1}^{k} a_i n_i$$

with $a_i \in \{1, \dots, d_i\}$ for $i \in \{1, \dots, k\}$ are mutually distinct $\mod n_1$.

2 Let $ABCD$ be a cyclic quadrilateral such that the triangles BCD and CDA are not equilateral. Prove that if the Simson line of A with respect to $\triangle BCD$ is perpendicular to the Euler line of BCD, then the Simson line of B with respect to $\triangle ACD$ is perpendicular to the Euler line of $\triangle ACD$.

3 Let A and B be finite sets of real numbers and let x be an element of $A + B$. Prove that

$$|A \cap (x - B)| \leq \frac{|A - B|^2}{|A + B|}$$

where $A+B = \{a+b : a \in A, b \in B\}$, $x-B = \{x-b : b \in B\}$ and $A-B = \{a-b : a \in A, b \in B\}$.

4 Prove that a finite simple planar graph has an orientation so that every vertex has out-degree at most 3.

5 Let p and q be two given positive integers. A set of $p + q$ real numbers $a_1 < a_2 < \cdots < a_{p+q}$ is said to be balanced iff $a_1, \dots, a_p$ were an arithmetic progression with common difference q and $a_p, \dots, a_{p+q}$ where an arithmetic progression with common difference p. Find the maximum possible number of balanced sets, so that any two of them have nonempty intersection.

Comment: The intended problem also had "p and q are coprime" in the hypothesis. A typo when the problems where written made it appear like that in the exam (as if it were the only typo in the olympiad). Fortunately, the problem can be solved even if we didn't suppose that and it can be further generalized: we may suppose that a balanced set has $m+n$ reals $a_1 < \cdots < a_{m+n-1}$ so that $a_1, \dots, a_m$ is an arithmetic progression with common difference p and $a_m, \dots, a_{m+n-1}$ is an arithmetic progression with common difference q.

Day 2

1 Prove that for any positive integer $n \geq 2$ we have that

$$\sum_{k=2}^{n} \lfloor \sqrt[k]{n} \rfloor = \sum_{k=2}^{n} \lfloor \log_k n \rfloor .$$

2 Let $ABCD$ be a convex circumscribed quadrilateral such that $\angle ABC + \angle ADC < 180^\circ$ and $\angle ABD + \angle ACB = \angle ACD + \angle ADB$. Prove that one of the diagonals of quadrilateral $ABCD$ passes through the other diagonals midpoint.

3 Find the maximum possible number of kings on a 12×12 chess table so that each king attacks exactly one of the other kings (a king attacks only the squares that have a common point with the square he sits on).

4 Let k be a positive integer. Find the maximum value of

$$a^{3k-1}b + b^{3k-1}c + c^{3k-1}a + k^2 a^k b^k c^k ,$$

where a, b, c are non-negative reals such that $a + b + c = 3k$.

Day 3

1 Let m and n be two positive integers greater than 1. Prove that there are m positive integers $N_1 , \ldots , N_m$ (some of them may be equal) such that

$$\sqrt{m} = \sum_{i=1}^{m} (\sqrt{N_i} - \sqrt{N_i - 1})^{\frac{1}{n}} .$$

2 Let γ be a circle and l a line in its plane. Let K be a point on l, located outside of γ. Let KA and KB be the tangents from K to γ, where A and B are distinct points on γ. Let P and Q be two points on γ. Lines PA and PB intersect line l in two points R and respectively S. Lines QR and QS intersect the second time circle γ in points C and D. Prove that the tangents from C and D to γ are concurrent on line l.

3 Let $a_1 , \ldots , a_n$ be positive integers and a a positive integer that is greater than 1 and is divisible by the product $a_1 a_2 \ldots a_n$. Prove that $a^{n+1} + a - 1$ is not divisible by the product $(a + a_1 - 1)(a + a_2 - 1) \ldots (a + a_n - 1)$.

4 Let S be a set of positive integers, each of them having exactly 100 digits in base 10 representation. An element of S is called *atom* if it is not divisible by the sum of any two (not necessarily distinct) elements of S. If S contains at most 10 atoms, at most how many elements can S have?

Day 4

1 Let ΔABC be a triangle. The internal bisectors of angles $\angle CAB$ and $\angle ABC$ intersect segments BC, respectively AC in D, respectively E. Prove that

$$DE \leq (3-2\sqrt{2})(AB+BC+CA).$$

2 Let $f,g:\mathbb{Z}\to[0,\infty)$ be two functions such that $f(n)=g(n)=0$ with the exception of finitely many integers n. Define $h:\mathbb{Z}\to[0,\infty)$ by

$$h(n)=\max\{f(n-k)g(k) : k\in\mathbb{Z}\}.$$

Let p and q be two positive reals such that $1/p+1/q=1$. Prove that

$$\sum_{n\in\mathbb{Z}} h(n) \geq \left(\sum_{n\in\mathbb{Z}} f(n)^p\right)^{1/p}\left(\sum_{n\in\mathbb{Z}} g(n)^q\right)^{1/q}.$$

3 Determine all finite sets S of points in the plane with the following property: if $x,y,x',y'\in S$ and the closed segments xy and $x'y'$ intersect in only one point, namely z, then $z\in S$.

Day 5

1 Find all triples (a,b,c) of positive integers with the following property: for every prime p, if n is a quadratic residue $\mod p$, then an^2+bn+c is a quadratic residue $\mod p$.

2 Let n be a positive integer. Find the value of the following sum

$$\sum_{(n)}\sum_{k=1}^{n} e_k 2^{e_1+\cdots+e_k-2k-n},$$

where $e_k\in\{0,1\}$ for $1\leq k\leq n$, and the sum $\sum_{(n)}$ is taken over all 2^n possible choices of $e_1,\ldots,e_n$.

3 Let m and n be two positive integers for which $m<n$. n distinct points $X_1,\ldots,X_n$ are in the interior of the unit disc and at least one of them is on its border. Prove that we can find m distinct points $X_{i_1},\ldots,X_{i_m}$ so that the distance between their center of gravity and the center of the circle is at least $\frac{1}{1+2m(1-1/n)}$.