Romania Team Selection Test 2013

www.artofproblemsolving.com/community/c4468

by iarnab_kundu, Drytime, ComplexPhi, lyukhson

Day 1

1 Given an integer $n \geq 2$, let a_n, b_n, c_n be integer numbers such that

$$\left(\sqrt[3]{2}-1\right)^n = a_n + b_n\sqrt[3]{2} + c_n\sqrt[3]{4}.$$

Prove that $c_n \equiv 1 \pmod 3$ if and only if $n \equiv 2 \pmod 3$.

2 Circles Ω and ω are tangent at a point P (ω lies inside Ω). A chord AB of Ω is tangent to ω at C; the line PC meets Ω again at Q. Chords QR and QS of Ω are tangent to ω. Let I, X, and Y be the incenters of the triangles APB, ARB, and ASB, respectively. Prove that $\angle PXI + \angle PYI = 90^\circ$.

3 Determine all injective functions defined on the set of positive integers into itself satisfying the following condition: If S is a finite set of positive integers such that $\sum\limits_{s\in S}\frac{1}{s}$ is an integer, then $\sum\limits_{s\in S}\frac{1}{f(s)}$ is also an integer.

4 Let n be an integer greater than 1. The set S of all diagonals of a $(4n-1)$-gon is partitioned into k sets, $S_1, S_2, \ldots, S_k$, so that, for every pair of distinct indices i and j, some diagonal in S_i crosses some diagonal in S_j; that is, the two diagonals share an interior point. Determine the largest possible value of k in terms of n.

Day 2

1 Suppose that a and b are two distinct positive real numbers such that $\lfloor na \rfloor$ divides $\lfloor nb \rfloor$ for any positive integer n. Prove that a and b are positive integers.

2 The vertices of two acute-angled triangles lie on the same circle. The Euler circle (nine-point circle) of one of the triangles passes through the midpoints of two sides of the other triangle. Prove that the triangles have the same Euler circle.

EDIT by pohoatza (in concordance with Luis' PS): Let ABC be a triangle with circumcenter Γ and nine-point center γ. Let X be a point on Γ and let Y, Z be on Γ so that the midpoints of segments XY and XZ are on γ. Prove that the midpoint of YZ is on γ.

3 Let S be the set of all rational numbers expressible in the form

$$\frac{(a_1^2+a_1-1)(a_2^2+a_2-1)\dots(a_n^2+a_n-1)}{(b_1^2+b_1-1)(b_2^2+b_2-1)\dots(b_n^2+b_n-1)}$$

for some positive integers $n, a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n$. Prove that there is an infinite number of primes in S.

---

4 Let k be a positive integer larger than 1. Build an infinite set $\mathcal{A}$ of subsets of $\mathbb{N}$ having the following properties:

(a) any k distinct sets of $\mathcal{A}$ have exactly one common element;
(b) any $k+1$ distinct sets of $\mathcal{A}$ have void intersection.

Day 3

---

1 Let a and b be two square-free, distinct natural numbers. Show that there exist $c>0$ such that

$$\left|\{n\sqrt{a}\}-\{n\sqrt{b}\}\right|>\frac{c}{n^3}$$

for every positive integer n.

---

2 Let γ a circle and P a point who lies outside the circle. Two arbitrary lines l and l' which pass through P intersect the circle at the points X, Y , respectively X', Y' , such that X lies between P and Y and X' lies between P and Y'. Prove that the line determined by the circumcentres of the triangles PXY' and $PX'Y$ passes through a fixed point.

---

3 Determine the largest natural number r with the property that among any five subsets with 500 elements of the set $\{1,2,\dots,1000\}$ there exist two of them which share at least r elements.

---

4 Let f and g be two nonzero polynomials with integer coefficients and $\deg f > \deg g$. Suppose that for infinitely many primes p the polynomial $pf+g$ has a rational root. Prove that f has a rational root.

Day 4

---

1 Fix a point O in the plane and an integer $n\geq 3$. Consider a finite family $\mathcal{D}$ of closed unit discs in the plane such that:
(a) No disc in $\mathcal{D}$ contains the point O; and
(b) For each positive integer $k<n$, the closed disc of radius $k+1$ centred at O contains the centres of at least k discs in $\mathcal{D}$.
Show that some line through O stabs at least $\frac{2}{\pi}\log\frac{n+1}{2}$ discs in $\mathcal{D}$.

---

2 Let n be an integer larger than 1 and let S be the set of n-element subsets of the set $\{1, 2, \ldots, 2n\}$. Determine

$$\max_{A\in S}\left(\min_{x,y\in A, x\neq y}[x,y]\right)$$

where $[x, y]$ is the least common multiple of the integers x, y.

---

3 Given an integer $n \geq 2$, determine all non-constant polynomials f with complex coefficients satisfying the condition

$$1 + f(X^n + 1) = f(X)^n.$$

Day 5

---

1 Let n be a positive integer and let x_1, . . ., x_n be positive real numbers. Show that:

$$\min\left(x_1, \frac{1}{x_1}+x_2, \cdots, \frac{1}{x_{n-1}}+x_n, \frac{1}{x_n}\right) \leq 2\cos\frac{\pi}{n+2} \leq \max\left(x_1, \frac{1}{x_1}+x_2, \cdots, \frac{1}{x_{n-1}}+x_n, \frac{1}{x_n}\right).$$

---

2 Let K be a convex quadrangle and let l be a line through the point of intersection of the diagonals of K. Show that the length of the segment of intersection $l \cap K$ does not exceed the length of (at least) one of the diagonals of K.

---

3 Given a positive integer n, consider a triangular array with entries a_{ij} where i ranges from 1 to n and j ranges from 1 to $n-i+1$. The entries of the array are all either 0 or 1, and, for all $i > 1$ and any associated j , a_{ij} is 0 if $a_{i-1,j} = a_{i-1,j+1}$, and a_{ij} is 1 otherwise. Let S denote the set of binary sequences of length n, and define a map $f\colon S \to S$ via $f\colon (a_{11}, a_{12}, \cdots, a_{1n}) \to (a_{n1}, a_{n-1,2}, \cdots, a_{1n})$. Determine the number of fixed points of f.
