German National Olympiad 2017, Final Round

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by Tintarn

– Day 1

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1 Given two real numbers p and q, we study the following system of equations with variables $x, y \in \mathbb{R}$:

$$x^2 + py + q = 0,$$
$$y^2 + px + q = 0.$$

Determine the number of distinct solutions (x, y) in terms of p and q.

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2 Let ABC be a triangle such that $|AB| \neq |AC|$. Prove that there exists a point $D \neq A$ on its circumcircle satisfying the following property:
For any points M, N outside the circumcircle on the rays AB and AC, respectively, satisfying $|BM| = |CN|$, the circumcircle of AMN passes through D.

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3 General Tilly and the Duke of Wallenstein play "Divide and rule!" (Divide et impera!).
To this end, they arrange N tin soldiers in M companies and command them by turns.
Both of them must give a command and execute it in their turn.

Only two commands are possible: The command "*Divide!*" chooses one company and divides it into two companies, where the commander is free to choose their size, the only condition being that both companies must contain at least one tin soldier.
On the other hand, the command "*Rule!*" removes exactly one tin soldier from each company.

The game is lost if in your turn you can't give a command without losing a company. Wallenstein starts to command.

a) Can he force Tilly to lose if they start with 7 companies of 7 tin soldiers each?

b) Who loses if they start with $M \geq 1$ companies consisting of $n_1 \geq 1, n_2 \geq 1, \dots, n_M \geq 1$ $(n_1 + n_2 + \dots + n_M = N)$ tin soldiers?

– Day 2

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4 Let $ABCD$ be a cyclic quadrilateral. The point P is chosen on the line AB such that the circle passing through C, D and P touches the line AB. Similarly, the point Q is chosen on the line CD such that the circle passing through A, B and Q touches the line CD.

Prove that the distance between P and the line CD equals the distance between Q and AB.

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5 Prove that for all non-negative numbers x, y, z satisfying $x + y + z = 1$, one has

$$1 \leq \frac{x}{1-yz} + \frac{y}{1-zx} + \frac{z}{1-xy} \leq \frac{9}{8}.$$

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6 Prove that there exist infinitely many positive integers m such that there exist m consecutive perfect squares with sum m^3. Specify one solution with $m > 1$.
