USAMO 1974

www.artofproblemsolving.com/community/c4472

by Brut3Forc3, rrusczyk

1 Let $a, b,$ and c denote three distinct integers, and let P denote a polynomial having integer coefficients. Show that it is impossible that $P(a) = b, P(b) = c,$ and $P(c) = a.$

---

2 Prove that if $a, b,$ and c are positive real numbers, then

$$a^a b^b c^c \geq (abc)^{(a+b+c)/3}.$$

---

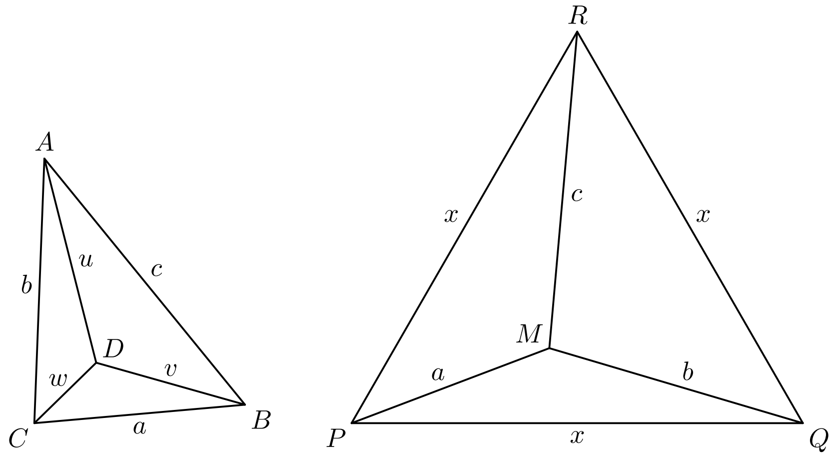
3 Two boundary points of a ball of radius 1 are joined by a curve contained in the ball and having length less than 2. Prove that the curve is contained entirely within some hemisphere of the given ball.

---

4 A father, a mother and son hold a family tournament, playing a two person board game with no ties. The tournament rules are:
(i) The weakest player chooses the first two contestants.
(ii) The winner of any game plays the next game against the person left out.
(iii) The first person to win two games wins the tournament.
The father is the weakest player, the son the strongest, and it is assumed that any player's probability of winning an individual game from another player does not change during the tournament. Prove that the father's optimal strategy for winning the tournament is to play the first game with his wife.

---

5 Consider the two triangles ABC and PQR shown below. In triangle $ABC, \angle ADB = \angle BDC = \angle CDA = 120^\circ$. Prove that $x = u + v + w$.


A
B
C
D
a
b
c
u
v
w
R
P
Q
M
x
x
x
a
b
c