USAMO 1977

www.artofproblemsolving.com/community/c4475

by Brut3Forc3, rrusczyk

1 Determine all pairs of positive integers (m, n) such that $(1 + x^n + x^{2n} + \cdots + x^{mn})$ is divisible by $(1 + x + x^2 + \cdots + x^m)$.

---

2 ABC and $A'B'C'$ are two triangles in the same plane such that the lines AA', BB', CC' are mutually parallel. Let $[ABC]$ denotes the area of triangle ABC with an appropriate $\pm$ sign, etc.; prove that

$$3([ABC] + [A'B'C']) = [AB'C'] + [BC'A'] + [CA'B'] + [A'BC] + [B'CA] + [C'AB].$$

---

3 If a and b are two of the roots of $x^4 + x^3 - 1 = 0$, prove that ab is a root of $x^6 + x^4 + x^3 - x^2 - 1 = 0$.

---

4 Prove that if the opposite sides of a skew (non-planar) quadrilateral are congruent, then the line joining the midpoints of the two diagonals is perpendicular to these diagonals, and conversely, if the line joining the midpoints of the two diagonals of a skew quadrilateral is perpendicular to these diagonals, then the opposite sides of the quadrilateral are congruent.

---

5 If a, b, c, d, e are positive numbers bounded by p and q, i.e, if they lie in $[p, q], 0 < p$, prove that

$$(a + b + c + d + e)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e}\right) \le 25 + 6\left(\sqrt{\frac{p}{q}} - \sqrt{\frac{q}{p}}\right)^2$$

and determine when there is equality.
