German National Olympiad 2015, Final Round

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– **Day 1**

1 Determine all pairs of real numbers (x, y) satisfying

$$x^3 + 9x^2y = 10,$$
$$y^3 + xy^2 = 2.$$

2 A positive integer n is called *smooth* if there exist integers $a_1, a_2, \ldots, a_n$ satisfying

$$a_1 + a_2 + \ldots + a_n = a_1 \cdot a_2 \cdot \ldots \cdot a_n = n.$$

Find all smooth numbers.

3 To prepare a stay abroad, students meet at a workshop including an excursion. To promote interaction between the students, they are to be distributed to two busses such that not too many of the students in the same bus know each other.

Every student knows all those who know her. The number of such pairwise acquaintances is k.

Prove that it is possible to distribute the students such that the number of pairwise acquaintances in each bus is at most $\frac{k}{3}$.

– **Day 2**

4 Let k be a positive integer. Define n_k to be the number with decimal representation $70\ldots01$ where there are exactly k zeroes. Prove the following assertions:

a) None of the numbers n_k is divisible by 13.
b) Infinitely many of the numbers n_k are divisible by 17.

5 Let $ABCD$ be a convex quadrilateral such that the circle with diameter AB touches the line CD. Prove that that the circle with diameter CD touches the line AB if and only if BC and AD are parallel.

6 Prove that for all $x, y, z > 0$, the inequality

$$\frac{x + y + z}{3} + \frac{3}{\frac{1}{x} + \frac{1}{y} + \frac{1}{z}} \geq 5\sqrt[3]{\frac{xyz}{16}}$$

holds. Determine if equality can hold and if so, in which cases it occurs.