USAMO 1980

www.artofproblemsolving.com/community/c4478

by Mrdavid445, rrusczyk

1 A two-pan balance is innacurate since its balance arms are of different lengths and its pans are of different weights. Three objects of different weights A, B, and C are each weighed separately. When placed on the left-hand pan, they are balanced by weights A_1, B_1, and C_1, respectively. When A and B are placed on the right-hand pan, they are balanced by A_2 and B_2, respectively. Determine the true weight of C in terms of A_1, B_1, C_1, A_2, and B_2.

---

2 Determine the maximum number of three-term arithmetic progressions which can be chosen from a sequence of n real numbers

$$a_1 < a_2 < \cdots < a_n.$$

---

3 Let $F_r = x^r \sin rA + y^r \sin rB + z^r \sin rC$, where x, y, z, A, B, C are real and $A + B + C$ is an integral multiple of π. Prove that if $F_1 = F_2 = 0$, then $F_r = 0$ for all positive integral r.

---

4 The inscribed sphere of a given tetrahedron touches all four faces of the tetrahedron at their respective centroids. Prove that the tetrahedron is regular.

---

5 Prove that for numbers a, b, c in the interval $[0, 1]$,

$$\frac{a}{b+c+1} + \frac{b}{c+a+1} + \frac{c}{a+b+1} + (1-a)(1-b)(1-c) \le 1.$$
