Belarusian National Olympiad 1997

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–	Grade IX
–	Day 1
1	$$Problem 1:$$ A two-digit number which is not a multiple of 10 is given. Assuming it is divisible by the sum of its digits, prove that it is also divisible by 3. Does the statement hold for three-digit numbers as well?
2	$$Problem 2:$$ Points D and E are taken on side CB of triangle ABC, with D between C and E, such that $\angle BAE = \angle CAD$. If $AC < AB$, prove that $AC.AE < AB.AD$.
3	$$Problem 3:$$ Is it possible to mark 10 red, 10 blue and 10 green points on a plane such that: For each red point A, the point (among the marked ones) closest to A is blue; for each blue point B, the point closest to B is green; and for each green point C, the point closest to C is red?
4	$$Problem 4:$$ The sum of 5 positive numbers equals 2. Let S_k be the sum of the $k-th$ powers of these numbers. Determine which of the numbers $2, S_2, S_3, S_4$ can be the greatest among them.
–	Day 2
1	$$Problem 1$$;Find all composite numbers n with the following property: For every proper divisor d of n (i.e. $1 < d < n$), it holds that $n - 12 \geq d \geq n - 20$.

2

$$Problem2:$$

If ABCD is as convex quadrilateral with $\angle ADC = 30$
and $BD = AB + BC + CA$,
prove that BD bisects $\angle ABC$.

3

$$Problem3;$$

If distinct real numbers x,y satisfy $\{x\} = \{y\}$ and $\{x^3\} = \{y^3\}$
prove that x is a root of a quadratic equation with integer coefficients.

4

$$Problem4$$

Straight lines k, l, m intersecting each other in three different points are drawn on a classboard. Bob remembers that in some coordinate system the linesk, l, m have the equations $y = ax, y = bx$ and $y = c + 2\frac{ab}{a+b}x$ (where $ab(a + b)$ is non zero).
Misfortunately, both axes are erased. Also, Bob remembers that there is missing a line n ($y = -ax + c$), but he has forgotten a, b, c. How can he reconstruct the line n?

– Grade XI

– Day 1

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1 Different points A_1, A_2, A_3, A_4, A_5 lie on a circle so that $A_1A_2 = A_2A_3 = A_3A_4 = A_4A_5$. Let A_6 be the diametrically opposite point to A_2, and A_7 be the intersection of A_1A_5 and A_3A_6. Prove that the lines A_1A_6 and A_4A_7 are perpendicular

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2 A sequence $(a_n)_{-\infty}^{-\infty}$ of zeros and ones is given. It is known that $a_n = 0$ if and only if $a_{n-6} + a_{n-5} + ... + a_{n-1}$ is a multiple of 3, and not all terms of the sequence are zero. Determine the maximum possible number of zeros among $a_0, a_1, ..., a_{97}$.

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3 Let $a, x, y, z > 0$. Prove that: $\frac{a+y}{a+z}x + \frac{a+z}{a+x}y + \frac{a+x}{a+y}z \geq x + y + z \geq \frac{a+z}{a+x}x + \frac{a+x}{a+y}y + \frac{a+y}{a+z}z$

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4 A set M consists of n elements. Find the greatest k for which there is a collection of k subsets of M such that for any subsets $A_1, ..., A_j$ from the collection, there is an element belonging to an odd number of them

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– Day 2

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1 We call the sum of any k of n given numbers (with distinct indices) a k-sum. Given n, find all k such that, whenever more than half of k-sums of numbers $a_1, a_2, ..., a_n$ are positive, the sum $a_1 + a_2 + ... + a_n$ is positive as well.

2 Suppose that a function $f : R^+ \to R^+$ satisfies

$$f(f(x)) + x = f(2x).$$

Prove that $f(x) \geq x$ for all $x > 0$

3 Here's the original problem:

Does there exist an infinite set M of straight lines on the coordinate plane
such that
(i) no two lines are parallel, and
(ii) for any integer point there is a line from M containing it?
Does the answer change if we add one more condition:
(iii) any line from M passes through at least 2 integer points?

4 A triangle $A_1B_1C_1$ is a parallel projection of a triangle ABC in space. The parallel projections A_1H_1 and C_1L_1 of the altitude AH and the bisector CL of $\triangle ABC$ respectively are drawn. Using a ruler and compass, construct a parallel projection of :
(a) the orthocenter,
(b) the incenter of $\triangle ABC$.